1. (10) Let \( z = 3 - 3\sqrt{3} i \) and \( w = -4 + 4i \). Write in rectangular form, \( a + bi \), each of the following:

   a. \( \frac{z^4}{w^2} \)

   b. \((z + 2\overline{w})(\overline{z} + 2w)\)

2. (10) Prove the following proposition: Let \((X, d)\) be a metric space. Let \( A, B \) be subsets of \( X \) such that \( A \subseteq B \). Then, \( \text{int} \ A \subseteq \text{int} \ B \).

3. (10) Let \( \{x_n\}, \{y_n\} \) be sequences in \( \mathbb{C} \) such that \( \lim x_n = x \) and \( \lim y_n = y \). Show that \( \lim x_n y_n \) exists and equals \( xy \).

4. (20) Prove the following proposition: Let \((X, d)\) be a metric space. A set \( F \subseteq X \) is closed if and only if for each sequence \( \{x_n\} \subseteq F \) with \( \lim x_n = x \) we have \( x \in F \).

   Note: This is Proposition 3.2 in Conway. It precedes Proposition 3.4 in Conway, which you did in your homework. (Proposition 3.4 states: a set is closed if and only if it contains all of its limit points.) Inasmuch as it would be circular to use Proposition 3.4 to prove Proposition 3.2, (since Proposition 3.2 is used to prove Proposition 3.4) do not give a proof for this proposition which employs Proposition 3.4.

5. (10) Prove the following theorem: Let \( G \) be an open subset of \( \mathbb{C} \). Then, the components of \( G \) are open.

6. (10) For \( z, w \in \mathbb{C} \), \( w \neq 0 \), show that \( \overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}} \).

7. (10) Provide a counterexample to each of the following assertions:

   a. In a metric space \((X, d)\), if a set \( A \) is closed, then \( A \) is compact.

   b. Let \((X, d), (\Omega, \rho)\) be metric spaces. Let \( f : X \to \Omega \). If \( f \) is continuous on \( X \), then \( f \) is uniformly continuous on \( X \).

   c. Let \((X, d)\) be a metric space. If \( X \) is complete, then \( X \) is compact.
d. Let \((X, d), (\Omega, \rho)\) be metric spaces. Let \(f_n : X \to \Omega, f : X \to \Omega\). If
\[f_n \to f\]on \(X\), then \(f_n \overset{\text{uniformly}}{\to} f\) on \(X\).

e. Let \((X, d), (\Omega, \rho)\) be metric spaces. Let \(f : X \to \Omega\). If \(f\) is continuous on \(X\) and if \(G\) is an open subset of \(X\), then \(f(G)\) is an open subset of \(\Omega\).

8. (20) Classification Problem. Correctly identify whether the following subsets of \(\mathbb{C}\) are: (a) open; (b) closed; (c) connected; (d) polygonally path connected; (e) compact; (f) complete; (g) bounded; (h) region. You do not need to provide a rationale for your classification. Fill out the classification information on the attach table.

A. \(\{z = x + iy : xy < 2, xy > 0\} \cap B(0, 4)\)

B. \(\{z = x + iy : 0 < y < e^x, -\infty < x < \infty\} \cap \{z : |\text{Im} z| < 2\}\)

C. \(\{z = x + iy : \frac{x^2}{25} + \frac{y^2}{4} \leq 1\} \setminus B(0, 3)\)

D. \(\overline{B(0, 1)} \setminus B(\mathbb{R}, \mathbb{C})\) \(\cap \{z : \text{Re} z > 0\}\)

E. \(\bigcup_{n=1}^{\infty} l_n\), where each \(l_n = [0, \text{cis} \left(\frac{\pi}{2n}\right)]\)

Classification Table for Problem 8

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