Solution Set #6

Section 7.10

1a. No. \( \int_0^1 \frac{1}{x^p} \, dx \) diverges if \( p > 1 \)

1b. Yes. \( \int_0^1 \frac{1}{x^p} \, dx \) converges if \( p < 1 \)

1c. Yes. \( \frac{x}{(16-x^4)^{1/3}} \leq \frac{2}{[(4+x^2)(2-x)]^{1/3}} \frac{1}{(2-x)^{1/3}} \leq \frac{2}{8^{1/3}} \frac{1}{(2-x)^{1/3}} \) for \( x \in [0,2] \) and
\[
\int_a^b \frac{1}{(b-x)^p} \, dx \text{ converges if } p < 1
\]

1e. Yes. \( \lim_{x \to 0^+} \frac{\log(1/x)}{\sqrt{x}} = 0 \) which implies there exists a constant \( A \) such that
\[
\frac{\log(1/x)}{\sqrt{x}} \leq A x^{3/4} \text{ for } x \in (0,1] \text{ and } \int_0^1 \frac{1}{x^p} \, dx \text{ converges if } p < 1
\]

1f. Yes. \( \sin x \leq x \) for \( x \in [0,1] \) which implies that \( \frac{\sin x}{x^{3/2}} \leq \frac{1}{x^{1/2}} \) for \( x \in [0,1] \) and
\[
\int_0^1 \frac{1}{x^p} \, dx \text{ converges if } p < 1
\]

3. \( \int_0^\infty \frac{x^{s-1}}{1+x} \, dx \) is convergent if and only if both \( \int_0^1 \frac{x^{s-1}}{1+x} \, dx \) and \( \int_1^\infty \frac{x^{s-1}}{1+x} \, dx \) are convergent. \( \int_0^1 \frac{x^{s-1}}{1+x} \, dx \) is convergent if and only if \( 1 - s < 1 \), i.e., \( 0 < s \). Since
\[
\frac{x^{s-1}}{1+x} \leq \frac{x^{s-1}}{x} = \frac{1}{x^{2-s}} \text{ for } x > 1,
\]
then \( \int_1^\infty \frac{x^{s-1}}{1+x} \, dx \) is convergent if and only if \( 2 - s > 1 \), i.e., \( s < 1 \). Therefore, \( \int_0^\infty \frac{x^{s-1}}{1+x} \, dx \) is convergent if and only if \( 0 < s < 1 \).

5a. Yes. \( \sin t \) is an odd function; therefore, the C.P.V. \( \int_{-\infty}^{\infty} \sin t \, dt = 0 \).

5b. No. \( |\sin t| \) is an even function and \( \int_{-\infty}^{\infty} |\sin t| \, dt \) diverges.
5c. Yes. \((1/(1+t^2))\) is an even function and \(\int_0^\infty \frac{1}{1+t^2} dt\) converges.

7. If \(f\) is continuous on \([0,1]\), then \(f\) attains a maximum on \([0,1]\), say \(M\). We have then that
\[
\frac{f(x)}{\sqrt{1-x^2}} \leq \frac{M}{\sqrt{1-x}} \leq M \frac{1}{\sqrt{1-x}}
\]
and \(\int_a^b \frac{1}{(b-x)^p} dx\) converges if \(p < 1\). Therefore, \(\int_0^1 \frac{f(x)}{\sqrt{1-x^2}} dx\) converges. We have
\[
\int_0^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \lim_{b \to 1^-} \int_0^b \frac{f(x)}{\sqrt{1-x^2}} dx.
\]
Letting \(x = \arcsin(u)\), then from Theorem 7.8G we have \(\int_0^{\arcsin(b)} f(\sin(u)) du\).

Therefore, we have
\[
\int_0^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \lim_{b \to 1^-} \int_0^b \frac{f(x)}{\sqrt{1-x^2}} dx = \lim_{b \to 1^-} \int_0^{\arcsin(b)} f(\sin(u)) du = \int_0^{\arcsin(b)} f(\sin(u)) du.
\]

Section 8.1

1a. \(\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = [\text{by (9) and (10)}] \frac{-\sinh(x)}{\cosh(x)} = -\tanh(x)\)

1b. By (8) \(C'(x) = S(x)\); hence, \(C''(x) = S'(x)\). Then, by (7) \(S'(x) = C(x)\). Therefore, \(C''(x) = C(x)\).

1c. By \(S'(x) = C(x)\) and \(C(x) > 0\) on \((-\infty, \infty)\). Theorem 7.7B implies that \(S(x)\) is strictly increasing on \((-\infty, \infty)\).

1d. Since by (5) \(S''(x) = S(x)\) on \((-\infty, \infty)\) and since \(S(0) = 0\), then \(1c\) (above) implies that \(S''(x) > 0\) for \(x > 0\) and \(S''(x) < 0\) for \(x < 0\). Hence, \(S\) is concave up for \(x > 0\) and \(S\) is concave down for \(x < 0\).

Section 8.2

1a. By (14) we have \(E(x) > 0\) and by (15) \(E'(x) > 0\). Thus, Theorem 7.7B implies that \(E(x)\) is strictly increasing on \((-\infty, \infty)\). By the comments following (14) we have that \(\lim_{x \to -\infty} E(x) = \infty\). By (12) \(E(-x) = 1/E(x)\). Hence, we must have
\[
\lim_{x \to -\infty} E(-x) = \lim_{x \to \infty} 1/E(x) = 0, \ i.e., \ \lim_{x \to \infty} E(x) = 0.
\]

1b. By (15) \(E'(x) = E(x)\) on \((-\infty, \infty); \ hence, \ E''(x) = E'(x)\) on \((-\infty, \infty)\). By (14) \(E(x) > 0\) on \((-\infty, \infty)\) which implies \(E''(x) > 0\) on \((-\infty, \infty)\). Hence, \(E(x)\) is concave on \((-\infty, \infty)\).
\[
\frac{E(x) + E(-x)}{2} = \frac{(C(x) + S(x)) + (C(-x) + S(-x))}{2} = \frac{2C(x)}{2} = C(x)
\]

\[
\frac{E(x) - E(-x)}{2} = \frac{(C(x) + S(x)) - (C(-x) + S(-x))}{2} = \frac{2S(x)}{2} = S(x)
\]

\[
\sinh(x) \cosh(y) + \cosh(x) \sinh(y) = \frac{e^{x+y} - e^{-x-y} + e^{x-y} - e^{-x+y}}{4} = \frac{e^{x+y} + e^{x-y} - e^{-x-y} - e^{-x+y}}{4}
\]

4a.\[
\frac{e^{x+y} + e^{x-y} - e^{-x-y} - e^{-x+y}}{4} + \frac{e^{x+y} - e^{x-y} + e^{-x-y} - e^{-x+y}}{4} = \frac{2e^{x+y} - 2e^{-(x+y)}}{4} = e^{x+y} - e^{-(x+y)} = \sinh(x+y)
\]

4c. By 4a (above) \(\sinh(2x) = \sinh(x + x) = \sinh(x) \cosh(x) + \cosh(x) \sinh(x) = 2 \sinh(x) \cosh(x)\).

4d. Using 4b \(\cosh(2x) = \cosh(x + x) = \cosh(x) \cosh(x) + \sinh(x) \sinh(x) = \cosh^2 x + \sinh^2 x\).

5a. \(\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}\)

5b. Because \(\tanh'(x) = \text{sech}^2(x) > 0\) on \((-\infty, \infty)\), we have that \(\tanh(x)\) is increasing on \((-\infty, \infty)\).

We have, from 5a (above) \(\lim_{x \to -\infty} \tanh x = \lim_{x \to -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1\). Similarly, \(\lim_{x \to \infty} \tanh x = -1\).

Hence, the range of \(\tanh x\) is \((-1, 1)\).

5c. Note that \(1 - \tanh^2 x = \text{sech}^2 x\). Let \(y = w(x)\) be the inverse function to \(\tanh\), i.e., let \(x = \tanh y\). Then, differentiating we have \(1 = (\text{sech}^2 y) y'\). Solving for \(y'\) we have \(y' = \frac{1}{\text{sech}^2 y} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2}\).
Section 8.3

1a. Applying Theorem 7.8A to the representation for \( L(x) \) in (23) we have \( L'(x) = 1/x \) for \( x \) in \((0, \infty)\). Then, Theorem 7.7B implies that \( L(x) \) is increasing for \( x \) in \((0, \infty)\).

1b. From 1a (above) we have \( L''(x) = -1/x^2 < 0 \) for \( x \) in \((0, \infty)\). Hence, \( L \) is concave down on \((0, \infty)\).

3a. \( f(x) = x^a = \exp(a \log x) \). Hence, \( f'(x) = \exp(a \log x) a (1/x) = a x^a / x = a x^{a-1} \).

3b. \( f(x) = a^x = \exp(x \log a) \). Hence, \( f'(x) = \exp(x \log a) (\log a) = a^x (\log a) \).

4. By the second mean value theorem for integrals (Theorem 8.5D) we have (for some \( c \) between 1 and \( x \) (in the case that \( x > 1 \))
\[
\int_1^x \frac{1}{t^{3/2}} \frac{1}{t^{1/2}} \frac{1}{t} \, dt = \frac{1}{c^{1/2}} \int_1^c \frac{1}{t} \, dt > \frac{1}{x^{1/2}} \int_1^x \frac{1}{t} \, dt = \frac{1}{\sqrt{x}} \log x
\]

Hence, \( \frac{1}{\sqrt{x}} \int_1^x \frac{1}{t^{3/2}} \, dt > \frac{\log x}{x} \) (\( > 0 \)) for \( x > 1 \). Hence, integrating the left-hand side we have \( \frac{1}{\sqrt{x}} \frac{2}{\sqrt{x}} \frac{\sqrt{x} - 1}{\sqrt{x}} > \frac{\log x}{x} > 0 \) for \( x > 1 \). But, then the squeeze theorem implies that as \( x \) tends to infinity that \( (\log x)/x \) tends to 0.

Section 8.4

1a. By (35) \( \sin(\pi/2 - x) = \sin(\pi/2)\cos(x) - \cos(\pi/2)\sin(x) \). But \( \sin(\pi/2) = 1 \) and \( \cos(\pi/2) = 0 \) (see remarks following line (29)). So, \( \sin(\pi/2 - x) = \cos(x) \).

1b. By (36) \( \cos(\pi/2 - x) = \cos(\pi/2)\cos(x) + \sin(\pi/2)\sin(x) \). But \( \sin(\pi/2) = 1 \) and \( \cos(\pi/2) = 0 \) (see remarks following line (29)). So, \( \cos(\pi/2 - x) = \sin(x) \).

2a. By (36) \( \cos(2x) = \cos(x + x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2 x - \sin^2 x \). But the later equals, by (34) \( (1 - \sin^2 x) - \sin^2 x \). Hence, \( \cos(2x) = 1 - 2 \sin^2 x \).

3a. By 2a (above) and the remarks following line (29) \( 0 = \cos(2(\pi/4)) = 1 - 2 \sin^2 \pi/4 \).

Hence, \( \sin \pi/4 = \pm \frac{1}{\sqrt{2}} \). Since \( \sin x > 0 \) for \( 0 < x < \pi/2 \), we have \( \sin \pi/4 = \frac{1}{\sqrt{2}} \). Mutatis mutandis, we have \( \cos \pi/4 = \frac{1}{\sqrt{2}} \).
4. According to line (28) we have \( \sin(x + 2\pi) = \sin(x + \pi + \pi) = -\sin(x + \pi) = -(-\sin x) = \sin x \). By line (30) and we have \( \cos(x) = \sin'(x) \). But the derivative of sine satisfies according to the remarks following line (30) the relation \( \sin'(x+\pi) = -\sin'(x) \). Hence, repeating the above argument we have \( \cos(x + 2\pi) = \cos(x) \).

7. By the quotient rule
\[
(tan x)' = \left( \frac{\sin x}{\cos x} \right)' = \frac{\cos x \cos x - \sin x (- \sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\]

Note that \( 1 + \tan^2 x = \sec^2 x \). Let \( y = w(x) \) be the inverse function to tan, i.e., let \( x = \tan^{-1} y \). Then, differentiating we have \( 1 = (\sec^2 y) y' \). Solving for \( y' \) we have \( y' = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2} \).