I. If $A$, $B$ and $C$ are sets, prove or disprove (by giving a counterexample) each of the following statements:

(a) (5) $A \cap B \subseteq A$

Proof. Let $x \in A \cap B$ then, by the definition of intersection of sets, $x \in A$ and $x \in B$. Thus, if $x \in A \cap B$ then $x \in A$. By the definition of subset, we have then that $A \cap B \subseteq A$.

(b) (5) $A \subseteq B$ implies $A \cap B = A$

Proof. We need to show that if $A \subseteq B$ then $A \cap B = A$. The latter is true if and only if both $A \cap B \subseteq A$ and $A \subseteq A \cap B$ are true. The part $A \cap B \subseteq A$ was proven in a), so we only need to prove that $A \subseteq A \cap B$. Let $x \in A$, then since by hypothesis $A \subseteq B$, we have that $x \in B$, that is $x \in A$ and $x \in B$. By definition of intersection, this means that $x \in A \cap B$, that is $A \subseteq A \cap B$.

(c) (5) If $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$ then $B \cap C \neq \emptyset$

This is not true, a counterexample can be given as follows: let $A = \{a, b, c\}$, $B = \{b\}$ and $C = \{c\}$, with $a \neq b$, $a \neq c$, $b \neq c$. Then $A \cap B = \{b\} \neq \emptyset$, $A \cap C = \{c\} \neq \emptyset$, however $B \cap C = \emptyset$ since $b \neq c$.

II. Let $S(n) = 1 + 2 + \cdots + n$. Using the principle of mathematical induction, prove that the following statement $P(n)$ is true for each positive integer $n$ (no credit will be given if you do not use the principle of mathematical induction):

$P(n) : \quad S(n) = \frac{n(n+1)}{2}$

Proof. $P(1)$ is true since $S(1) = 1 = \frac{1(1+1)}{2}$

We then need to show that if $P(k)$ is true then $P(k+1)$ is true as well, that is, if $S(k) = \frac{k(k+1)}{2}$ then $S(k+1) = \frac{(k+1)(k+2)}{2}$. So we assume that $S(k) = \frac{k(k+1)}{2}$. We have, $S(k+1) = 1+2+\cdots+k+(k+1) = S(k) + (k+1) = \frac{k(k+1)}{2} + (k+1)$, by hypothesis. So, $S(k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$.
III. Write the negation of the following

**Statement:** “For each \( x \), \( e^x > 0 \)

**Negation:** “There is one \( x \) such that \( e^x \leq 0 \)

IV. Write the converse of the following

**Statement:** “If \( n \) is a nonnegative integer then \( n \) is a rational number.”

**Converse:** “If \( n \) is a rational number then \( n \) is a nonnegative integer.”

Is the converse true? Prove it if true or give a counterexample otherwise.

Not true, for example \( n = \frac{1}{2} \) is a rational number which is not an integer.

V. Write the contrapositive of the following

**Statement:** “If \( A \) is a subset of \( B \) then \( A \cap B = A \).”

**Contrapositive:** “If \( A \cap B \neq A \) then \( A \) is not a subset of \( B \).”

VI. Decide which of the following statements is logically equivalent to

“If \( y \) is an integer and \( y \) is a square then \( y \) is a positive integer”

Circle your choice, you do not need to justify.

a) If \( y \) is not an integer and \( y \) is not a square then \( y \) is not a positive integer.

b) If \( y \) is a positive integer then \( y \) is an integer and \( y \) is a square.

c) If \( y \) is not a positive integer then \( y \) is not an integer or \( y \) is not a square.

**Ans.** c) is the correct one.

VII. Prove that if \( \beta : S \to T, \gamma : S \to T, \alpha : T \to U \), \( \alpha \) is one-to-one, and \( \alpha \circ \beta = \alpha \circ \gamma \) then \( \beta = \gamma \).

**Proof.** We need to show that, under the hypothesis, \( \beta = \gamma \). By the definition of when two mappings are equal, this is equivalent to showing that \( \beta(x) = \gamma(x) \) for all \( x \in S \), since we already know that they have same domain and codomain.

So, let \( x \in S \). Since \( \alpha \circ \beta = \alpha \circ \gamma \), we have \( (\alpha \circ \beta)(x) = (\alpha \circ \gamma)(x) \). That is, \( \alpha(\beta(x)) = \alpha(\gamma(x)) \), by the definition of composition of functions. Now, since \( \alpha \) is one-to-one and \( \alpha(\beta(x)) = \alpha(\gamma(x)) \), for \( x \in S \) it has to be \( \beta(x) = \gamma(x) \), for \( x \in S \). This concludes the proof.

□
a)(3) Prove that the set of even positive integers

\[ A = \{ n \in \mathbb{N} : n = 2m, \text{ for some } m \in \mathbb{N} \} \]

is closed under +.

**Proof.** Let \( n_1, n_2 \in A \) then there are \( m_1, m_2 \in \mathbb{N} \) such that \( n_1 = 2m_1 \) and \( n_2 = 2m_2 \). Therefore, \( n_1 + n_2 = 2m_1 + 2m_2 = 2(m_1 + m_2) \), so \( n_1 + n_2 \in A \), being an even positive integer itself. \( \square \)

b)(2) Prove that the set of odd positive integers

\[ B = \{ n \in \mathbb{N} : n = 2k + 1, \text{ for some } k \in \mathbb{N} \} \]

is not closed under +. You can prove it either by exhibiting a specific counterexample or in general. Either, for instance, \( 5 + 7 = 12 \neq 2k + 1 \) for all \( k \in \mathbb{N} \) or, if \( n_1 = 2k_1 + 1, n_2 = 2k_2 + 1 \), with \( k_1, k_2 \in \mathbb{N} \) then \( n_1 + n_2 = 2(k_1 + k_2) + 2 = 2(k_1 + k_2 + 1) \) so \( n_1 + n_2 \notin A \) as it is not an odd positive integer.