For each of the following linear systems use Gaussian elimination to find all the solutions or to show that the system is inconsistent. If it is inconsistent explain why.

\begin{align*}
    x_2 + x_3 - 2x_4 - 2x_5 &= -3 \\
    x_1 + 2x_2 - x_3 &= 2 \\
    2x_1 + 4x_2 + x_3 - 3x_4 - 3x_5 &= -2 \\
    x_1 - 4x_2 - 7x_3 - x_4 - x_5 &= -19
\end{align*}

\begin{align*}
    x_1 - 2x + 2x_3 &= 4 \\
    x_1 + x_3 &= 6 \\
    2x_1 - 3x_2 + 5x_3 &= 4 \\
    3x_1 + 2x_2 - x_3 &= 1
\end{align*}
Answer the following questions:

a) What is an elementary matrix of order $n$?

b) What does it mean that a matrix $A$ has an inverse?

c) What does it mean that two $n \times n$ matrices $A$ and $B$ are row equivalent?

d) If a matrix $A$ is row equivalent to a matrix $B$ and $B$ is row equivalent to a matrix $C$, what can be said about $A$ and $C$? Justify your answer with a rigorous proof.

e) Is a matrix $A$ row equivalent to itself? Justify your answer with a rigorous proof.

f) If $A$ is an $n \times n$ matrix and $\alpha$ is a scalar, then $\det(\alpha A) = \alpha^n \det(A)$
III. (5) Given $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 5 & 4 \end{pmatrix}$, compute the following matrices if possible. When not possible, indicate so and justify your answer.

First, write $A^T = \begin{pmatrix} \end{pmatrix}$, $B^T = \begin{pmatrix} \end{pmatrix}$ and $I = \begin{pmatrix} \end{pmatrix}$, where $I$ denotes the $3 \times 3$-identity matrix.

a) $A + I$

b) $B^T A$

c) $A^T B^T$

d) $B^2$

e) $BB^T$
IV. For the matrix
\[
A = \begin{bmatrix}
1 & 2 & 3 \\
2 & 5 & 6 \\
1 & 3 & 4
\end{bmatrix}
\]
a) Find elementary matrices $E_1$, $E_2$ and $E_3$ such that $E_3E_2E_1A = U$ is an upper triangular matrix.
b) Find the inverses of the matrices $E_1$, $E_2$ and $E_3$.
c) Find a lower triangular matrix $L$ such that $A = LU$, where $U$ is the matrix found in part a).
Find the inverse of the matrix

\[
A = \begin{pmatrix}
1 & 3 & -12 \\
-2 & -1 & 6 \\
-1 & 0 & 1
\end{pmatrix}
\]
VI. Use Gaussian elimination to find the \( \text{det}(A) \) where

\[
A = \begin{pmatrix}
  2 & 0 & 0 & 1 \\
  0 & 1 & 3 & -3 \\
 -2 & -3 & -5 & 2 \\
  4 & -4 & 4 & -6 \\
\end{pmatrix}
\]