ON GENERIC DIFFERENTIAL $\text{SO}_n$-EXTENSIONS

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Abstract. Let $\mathcal{C}$ be an algebraically closed field with trivial derivation and let $\mathcal{F}$ denote the differential rational field $\mathcal{C}(Y_{ij})$, with $Y_{ij}$, $1 \leq i \leq n - 1$, $1 \leq j \leq n$, $i \leq j$, differentially independent over $\mathcal{C}$. We show that there is a Picard-Vessiot extension $\mathcal{E} \supset \mathcal{F}$ for a matrix equation $X' = XA(Y_{ij})$, with differential Galois group $\text{SO}_n$, with the property that if $F$ is any differential field with field of constants $\mathcal{C}$ then there is a Picard-Vessiot extension $E \supset F$ with differential Galois group $H \leq \text{SO}_n$ if and only if there are $f_{ij} \in F$ with $A(f_{ij})$ well defined and the equation $X' = XA(f_{ij})$ giving rise to the extension $E \supset F$.

1. Introduction

Let $\mathcal{C}$ denote an algebraically closed field with trivial derivation, $G$ a linear algebraic group over $\mathcal{C}$, and $\mathfrak{gl}_m(\cdot)$ the Lie algebra of $m \times m$ matrices with coefficients in some specified field. The short form ‘Picard-Vessiot $G$-extension’ (or some times ‘PVE with group $G$’) will be used for ‘Picard-Vessiot extension (PVE) with differential Galois group isomorphic to $G$’. We consider the differential rational field $\mathcal{F} = \mathcal{C}(Z_1, \ldots, Z_k)$, where $Z_1, \ldots, Z_k$ are differentially independent over $\mathcal{C}$.

Definition 1.1. A Picard-Vessiot $G$-extension $\mathcal{E} \supset \mathcal{F}$ for the equation $X' = XA(Z_1, \ldots, Z_k)$, with $A(Z_1, \ldots, Z_k) \in \mathfrak{gl}_m(\mathcal{F})$ for some $m$, is said to be a generic extension for $G$ if for every Picard-Vessiot $G$-extension $E \supset \mathcal{F}$ there is a specialization $Z_i \to f_i \in F$, such that the equation $X' = XA(f_1, \ldots, f_k)$ gives rise to $E \supset F$ and any fundamental solution matrix maps to one for the specialized equation.

Note that by making the assumption that $G = G(\mathcal{C})$, we are also assuming that the base field of a Picard-Vessiot $G$-extension, and the extension itself, have field of constants $\mathcal{C}$.

In this paper we produce generic extensions for the special orthogonal groups $\text{SO}_n$, $n \geq 3$. For $n = 2$ the group is isomorphic to the (cohomologically trivial) multiplicative group, a case already studied in [5].

The construction that we provide is based on Kolchin’s Structure Theorem, which describes the possible Picard-Vessiot $G$-extensions of a differential field $F$ as function fields of $F$-irreducible $G$-torsors [11, Theorem 5.12], [12, Theorem 1.28]. The isomorphism classes of $G$-torsors, in turn, are in bijective correspondence with the elements of the first Galois cohomology.
set $H^1(F, G)$ \cite{13, 15}. The latter is a particularly convenient feature since for the special orthogonal groups the first cohomology can be described in terms of regular quadratic forms of discriminant 1 (cf. \cite{7}).

In previous work the first author has studied generic extensions in two special situations. The first was when $G$ is connected and the extension is the function field of the trivial $G$-torsor (cf. \cite{5}). The second was when $G$ is the semidirect product $H \rtimes G^0$ of its connected component by a finite group $H$ and the extensions are the function fields of $F$-irreducible $G$-torsors of the form $W \times G^0$, where $W$ is an $F$-irreducible $H$-torsor (cf. \cite{6}).

In the present paper we turn our attention to the general case, that is, when $H^1(F, G)$ is not necessarily trivial. In \cite{7} we showed that in such a situation, it might be possible to find a Picard-Vessiot $G$-extension of $F$ that is the function field of a non-trivial torsor. We will use the machinery developed there and a version of a method to construct generic extensions from \cite{5} to attack this general situation when $G$ is the special orthogonal group $SO_n$, $n \geq 3$. With the description of the $SO_n$-torsors in terms of regular quadratic forms of discriminant 1 at our disposal we can provide a good description of the twisted Lie algebras associated to the torsors \cite{7}, a key ingredient of our construction.

Having a good grasp of the torsors also allows us to show that this extension fully descends to subgroups of $SO_n$, that is, there is a specialization of the parameters over the base field $F$ yielding a Picard-Vessiot $H$-extension if and only if $H \leq SO_n$.

Finally, we discuss how to proceed with connected groups in general, when a good description of the torsors is not available. In this case a generic extension relative to the trivial torsor along with the Trivialization Lemma from Section 3 allow a (not so explicit but quite similar) construction in which the specialization of the parameters takes place over a finite extension of $F$ instead of $F$.

All the differential fields that we consider are of characteristic zero and have algebraically closed field of constants. We keep the notations $C$ and $F$ introduced above.

2. Generic extension vs. generic equation

The $SO_n$ case is included among the groups studied by Goldman \cite{3} and Bhandari-Sankaran \cite{1}.

**Definition 2.1.** (Goldman \cite{3}) Let $G$ be a linear algebraic group over $\mathcal{C}$ and assume that a faithful representation in $GL_n(\mathcal{C})$ is given. Let $L(t, y) = Q_0(t_1, \ldots, t_r)y(\nu) + \cdots + Q_n(t_1, \ldots, t_r)y \in \mathcal{C}\{t_1, \ldots, t_r, y\}$ and write $(\pi_1, \ldots, \pi_n)$ for a fundamental system of zeros of $L(t, y)$ such that $\mathcal{C}\{t_1, \ldots, t_r, \pi_1, \ldots, \pi_n\}$ is a PVE of $\mathcal{C}\{t_1, \ldots, t_r\}$ with group $G$. Then $L(t, y) = 0$ will be called a generic equation with group $G$ if:

1. $t_1, \ldots, t_r$ are differentially independent over $\mathcal{C}$, and $\mathcal{C}\{t_1, \ldots, t_r\} \subset \mathcal{C}\{\pi_1, \ldots, \pi_n\}$.
(2) For every specialization \((t_1, \ldots, t_r, \pi_1, \ldots, \pi_n) \rightarrow (\bar{t}_1, \ldots, \bar{t}_r, \bar{\pi}_1, \ldots, \bar{\pi}_n)\) over \(C\) such that \(C\{\bar{t}_1, \ldots, \bar{t}_r, \bar{\pi}_1, \ldots, \bar{\pi}_n\}\) is a PVE of \(C\{\bar{t}_1, \ldots, \bar{t}_r\}\) and the field of constants of the latter is \(C\), the differential Galois group of this extension is a subgroup of \(G\).

(3) If \((\omega_1, \ldots, \omega_n)\) is a fundamental system of zeros of \(L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y \in F\{y\}\), where \(F\) is any differential field with field of constants \(C\), and \(F\{\omega_1, \ldots, \omega_n\}\) is a PVE of \(F\) with differential Galois group \(H \leq G\), then there exists a specialization \((t_1, \ldots, t_r) \rightarrow (\bar{t}_1, \ldots, \bar{t}_r)\) over \(F\) with \(\bar{t}_i \in F\) such that \(Q_o(\bar{t}_1, \ldots, \bar{t}_r) \neq 0\) and

\[a_i = Q_i(\bar{t}_1, \ldots, \bar{t}_r)Q_o^{-1}(\bar{t}_1, \ldots, \bar{t}_r)\]

Goldman shows that a necessary condition for such an equation to exist is that the number of parameters \(r\) equals the order \(n\) of the equation [3, Lemma 1, p. 343]. The groups studied in that paper include \(GL_n\), \(SL_n\) as well as the orthogonal and symplectic groups.

Now, let \(G\) act on \(C\{y_1, \ldots, y_n\}\), where \(y_1, \ldots, y_n\) are differentially independent over \(C\), by \(\sigma(y_i) = \sum_{j=1}^n c_{ij} y_j\) for \(\sigma = (c_{ij}) \in G(C) \subset GL_n(C)\). Then

\[P_i = \frac{W_i(y_1, \ldots, y_n)}{W_0(y_1, \ldots, y_n)} \quad (i = 1, \ldots, n),\]

where

\[W_i = (-1)^i \begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{(n-i+1)} & \cdots & y_n^{(n-i+1)} \end{vmatrix},\]

are invariant under the \(G\) action.

The procedure used by Goldman for the groups above first finds \(n\) differentially independent generators \(t_1, \ldots, t_n\) over \(C\) of the fixed field \(C\{y_1, \ldots, y_n\}^G\) and \(n+1\) differential polynomials \(Q_0(t_1, \ldots, t_n), \ldots, Q_n(t_1, \ldots, t_n) \in C\{t_1, \ldots, t_n\}\) with

\[P_i = \frac{Q_i(t_1, \ldots, t_n)}{Q_0(t_1, \ldots, t_n)} \quad (i = 1, \ldots, n).\]

He then shows that a generic equation with group \(G\) is given by

\[L(t, y) = Q_0(t_1, \ldots, t_n)y^{(n)} + \cdots + Q_n(t_1, \ldots, t_n)y = 0. \quad (1)\]

This method, however, fails to produce a generic equation for \(G = SO_3\) as [3, Example 3, p. 355] illustrates.

Bhandari and Sankaran [1] proved that (1) is generic for the special orthogonal groups in a weaker sense, that is, replacing (3) in Goldman’s definition with the following:
(3') If $F$ is a differential field with field of constants $C$ and $E$ is a PVE of $F$ with differential Galois group $H \leq G$, then there exists a linear differential equation

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0,$$

such that $Q_o(t_1, \ldots, t_r) \neq 0$, $a_i = Q_i(t_1, \ldots, t_r)Q_o^{-1}(t_1, \ldots, \bar{t}_r)$, $i = 1, \ldots, n$, for suitable $\bar{t}_i \in F$ and $E = F(\omega_1, \ldots, \omega_n)$ for a fundamental system of zeros of $L(y)$. There are, however, some key differences in our approaches. In constructing their equations, both Goldman and Bhandari-Sankaran start with the differential rational field $\mathcal{F} = C(y_1, \ldots, y_n)$, where $n$ is the order of the equation, and find the differential fixed field $C(y_1, \ldots, y_n)^G$. We start instead with $\mathcal{F}$ as our base field and show that $\mathcal{F}(Y)$, where $Y$ is a generic point of a “general” $G$-torsor, is a generic PVE in the sense of Definition 1.1. Furthermore, it satisfies descent conditions analogous to (2) and (3') above. In our case, the number $n$ of parameters is given by the dimension of the group and the description of the torsors, so it is independent of the representation of $G$ in a $GL_m$. By using a general derivation in the function field of our special $G$-torsor (that is, a typical element in the twisted Lie algebra) the specialization of our parameters comes in a very natural and painless fashion, whereas in the case of the generic equations in [1, 3], showing that $Q_0(t_1, \ldots, t_n) \neq 0$ is quite involved.

In connection with the previous notions of generic equation [1, 3] Juan-Magid [8] study the ring of generic solutions for a linear monic order $n$ equation, that is, $\mathcal{R} = \mathcal{C}\{P_1, \ldots, P_n\} \otimes _\mathcal{C} \mathcal{C}[y_i^{(j)}, 1 \leq i \leq n, 0 \leq j \leq n-1][w_0^{-1}]$, where $P_i$, $y_i$, $1 \leq i \leq n$, and $w_0$, are as above, with the $GL_n(\mathcal{C})$ action extended from the linear action on $V = \mathcal{C}y_1 + \cdots + \mathcal{C}y_n$ using the $\mathcal{C}$-basis $y_1, \ldots, y_n$. The ring $\mathcal{R}$ has the following properties:

Assume that $E \supset F$ is a Picard-Vessiot $G$-extension and that $G$ has a faithful representation $\rho$ in $GL_n$. Then there is a differential homomorphism $\Psi : R \to F$ such that

1. $E$ is the quotient field of $F^\Psi(\mathcal{R})$; and
2. $E \supset F$ is a PVE for

$$L(Y) = Y^{(n)} + \Psi(P_1)Y^{(n-1)} + \cdots + \Psi(P_n)Y^{(0)}$$

3. $\Psi$ is $G$-equivariant, so $\Psi(\mathcal{R}^G) \subset E^G = F$.

Conversely, assume that $G$ is an observable subgroup of $GL_n$ and let $\phi : \mathcal{R}^G \to F$ be a differential $F$-algebra homomorphism with restriction $\alpha$ to $\mathcal{R}^{GL_n}$. Let $P$ be a maximal differential ideal of $R = F \otimes \mathcal{R}$ whose inverse image in $\mathcal{R}$ contains the kernel of $\phi$, and let $E$ be the fraction field of $R/P$. Then $E$ is a PVE of $F$ with differential Galois group contained in $G$.

The special orthogonal groups are observable (see [4]) and therefore satisfy the above conditions. We point out that in our construction the coordinate ring $\mathcal{C}\{Y_{ij}\}[Y, 1/\det(Y)]$, where $Y$ is a generic point of a general $SO_n$-torsor, has properties similar to that of the ring $\mathcal{R}$. 
The work in [1, 3, 8] describes equations given by linear differential operators attached to a representation of the differential Galois group \( G \) in \( \text{GL}_n \). Our work describes matrix equations with group \( G \) in connection with the structure of the Picard-Vessiot \( G \)-extensions.

### 3. \( \text{SO}_n \)-extensions

In [7] we saw that every \( F \)-irreducible \( \text{SO}_n \)-torsor has a generic point of the form \( Y = XP \), where \( X \) is a generic point for \( \text{SO}_n \) and

\[
P = \begin{pmatrix}
\sqrt{a_1} & & \\
& \sqrt{a_2} & \\
& & \ddots \\
& & & \sqrt{a_n}
\end{pmatrix},
\]

for \( a_j \in F^* \) with \( a_1 \cdots a_n = 1 \) and the roots chosen to have product 1 as well. A PVE of \( F \) with group \( \text{SO}_n \) corresponding to this torsor, if any, equals the function field \( F(Y) \) of the torsor and has derivation given by \( Y' = YB \), where the matrix \( B \) is of the form

\[
\begin{pmatrix}
\frac{a_i'}{2a_n} & b_{12} & b_{13} & \cdots & b_{1n} \\
-a_2 b_{12} & \frac{a_2'}{2a_3} & b_{23} & \cdots & b_{2n} \\
-\frac{a_3}{a_3} b_{13} & -\frac{a_3'}{2a_3} b_{23} & \frac{a_3'}{2a_n} & \cdots & b_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{a_n}{a_n} b_{1n} & -\frac{a_n'}{a_n} b_{2n} & -\frac{a_n'}{a_n} b_{3n} & \cdots & \frac{a_n'}{2a_n}
\end{pmatrix}
\]

for \( b_{ij} \in F, \ 1 \leq i \leq n-1, \ 2 \leq j \leq n \) and \( a_i \in F^* \) as before. An explicit example was given there, with \( Y \) corresponding to a non-trivial torsor, by making the simplifying assumption that \( b_{i,i+1} = a_i \). We point out that with that assumption, the number of parameters used in [7] to produce a PVE associated to a non-trivial torsor is \( \frac{1}{2}n(n-1) \), the dimension of \( \text{SO}_n \).

Since our goal here is to produce a generic extension we need to modify that example in order to retain the \( \frac{1}{2}(n-1)(n+2) \) parameters in the matrix \( B \).

We assume that \( a_1, \ldots, a_{n-1}, b_{12}, \ldots, b_{n-1,n} \) are differentially independent over \( \mathcal{C} \) and let \( \mathcal{F} = \mathcal{C}(a_1, \ldots, a_{n-1}, b_{12}, \ldots, b_{n-1,n}) \). We first show that the equation \( \eta' = \eta A \) over the algebraic closure \( \overline{\mathcal{F}} \) of \( \mathcal{F} \), with coefficient matrix
the differential field corresponding equation has differential Galois group $SO_n$. Therefore, the field of constants of $\bar{F}$ which by [7, Lemma 1] is non-trivial.

The discussion in [7, Section 4] implies that the matrix

$$A = \begin{pmatrix}
0 & \sqrt{a_1}/a_2 & \sqrt{a_1}/a_3 & \ldots & \sqrt{a_1}/a_n \\
-\sqrt{a_1}/a_2 & 0 & \sqrt{a_1}/a_3 & \ldots & \sqrt{a_1}/a_n \\
-\sqrt{a_1}/a_3 & -\sqrt{a_1}/a_2 & 0 & \ldots & \sqrt{a_1}/a_n \\
& & & \ddots & \\
-\sqrt{a_1}/a_n & -\sqrt{a_1}/a_2 & -\sqrt{a_1}/a_3 & \ldots & 0 \\
\end{pmatrix}$$

has differential Galois group $SO_n$. From this it will follow that the corresponding equation $\eta' = \eta B$ over $\bar{F}$ has the same group.

Let $Z_{ij} = \sqrt{a_i}/\sqrt{a_j b_{ij}}, 1 \leq i \leq n - 1, 2 \leq j \leq n, i < j$. Clearly the $Z_{ij}$ are differentially independent over $\mathcal{C}$ since all the $a_i$ and $b_{ij}^2$ are in the differential field $\mathcal{C} = \mathcal{C}(a_1, \ldots, a_{n-1}, Z_{12}, \ldots, Z_{n-1,n})$, which forces the differential transcendence degree [10, Definition 3.2.33 and Theorem 5.4.12] of $\mathcal{C}$ over $\mathcal{L}$ to be $\frac{1}{2}(n - 1)(n + 2)$.

Now, since $A = \sum_{j=i+2}^n \sum_{j=1}^{n-1} Z_{ij} A_{ij}$, where $\{A_{ij}\}$ is the basis of $\text{Lie}(SO_n)$ consisting of the antisymmetric matrices with 1 in the $ij$-entry, $-1$ in the $ji$-entry and 0 otherwise, by [5, Theorem 4.1.2] it then follows that $\mathcal{L}(SO_n) \supset \mathcal{L}$, is a PVE with group $SO_n$ for the equation $X' = XA$.

Since $a_i, b_{ij}^2 \in \mathcal{L}$ we have that $a_i, b_{ij} \in \mathcal{L}$ and thus $\bar{F} = \bar{\mathcal{L}}$. Therefore, $\bar{F}(SO_n) \supset \bar{\mathcal{L}}(SO_n)$ is an algebraic extension. Since the field of constants of $\mathcal{L}(SO_n)$ is the algebraically closed field $\mathcal{C}$, $\bar{F}(SO_n)$ must have no new constants and $\bar{F}(SO_n) \supset F$ is a PVE with group $SO_n$.

The discussion in [7, Section 4] implies that the matrix

$$B = \begin{pmatrix}
\frac{a'_1}{2a_1} & b_{12} & b_{13} & \ldots & b_{1n} \\
-\frac{a_2}{a_1} b_{12} & \frac{a'_2}{2a_2} & b_{23} & \ldots & b_{2n} \\
-\frac{a_3}{a_1} b_{13} & -\frac{a_2}{a_3} b_{23} & \frac{a'_3}{2a_3} & \ldots & b_{3n} \\
& & & \ddots & \\
-\frac{a_n}{a_1} b_{1n} & -\frac{a_2}{a_n} b_{2n} & -\frac{a_3}{a_n} b_{3n} & \ldots & \frac{a'_n}{2a_n} \\
\end{pmatrix}$$

defines a derivation on the coordinate ring $T = F[Y]$ of the $SO_n$-torsor corresponding to the quadratic form given by the matrix

$$Q = \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix}$$

which by [7, Lemma 1] is non-trivial.

Since $\bar{F}(Y) = \bar{F}(X)$, as a differential field it will be isomorphic to $\bar{F}(SO_n)$. Therefore, the field of constants of $\bar{F}(Y)$ is $\mathcal{C}$. In particular, this implies that $\bar{F}(Y) \supset F$ is a no new constant extension. This shows that the function
field of the (non-trivial) $\text{SO}_n$-torsor corresponding to $Y$ is a PVE of $\mathcal{F}$ with group $\text{SO}_n$.

We point out for later use that the previous argument can be shown in a more general setting:

**Trivialization Lemma.** Let $E \supset F$ be a Picard-Vessiot $G$-extension with $G$ connected. Then there are a finite extension $k \supset F$ and a Picard-Vessiot $G$-extension $K = kE$ of $k$, such that $K = k(G)$.

In other words, if there is a PVE of $F$ with group $G$ then the trivial $G$-torsor can be realized over a finite extension of $F$. Although this is a known result (see [14, p. 142, Corollary]), for the convenience of the reader we include a short proof using the tools that we develop here.

**Proof.** Let $X$ be a generic point of $G$. Then $E = F(Y)$ where $Y = XP$, for a matrix $P$ with coefficients in $\mathcal{F}$ [7, Section 3]. Let $k$ denote the field generated over $F$ by the entries of $P$. Then $k(X) = k(Y) \supset F(Y)$ is an algebraic extension. Therefore, $k(G) = k(X) \supset k$ is a new constant extension and thus a Picard-Vessiot $G$-extension. Clearly, $K = k(X) = kE$. □

4. **Generic Extensions**

First we introduce the following notion, analogous to one for generic polynomial equations (see Kemper [9]).

**Definition 4.1.** A generic extension $E \supset \mathcal{F}$ for $G$ is called descent generic when the following condition holds: for any differential field $F$ with field of constants $\mathcal{C}$ there is a PVE $E \supset F$ with group $H \leq G$ if and only if there are $f_i \in F$ such that the matrix $A(f_1, \ldots, f_k)$ is well defined and the equation $X' = XA(f_1, \ldots, f_k)$ gives rise to the extension $E \supset F$.

**Theorem 1.** The extension $\mathcal{F}(Y) \supset \mathcal{F}$ is a generic PVE for $\text{SO}_n$. Furthermore, it is descent generic.

**Proof.** For convenience, we will use the double subscript notation $Y_{ii}$ for $a_i$, $i = 1, \ldots, n-1$, and put $Y_{ij} = b_{ij}$, $i < j$. We then let $A(Y_{ij}) = B$.

Suppose that $E \supset F$ is a PVE with group $H \leq \text{SO}_n$. Let $X, X_H$ respectively denote generic points of $\text{SO}_n$ and $H$. Then $E = F(Y)$, where $Y = X_H P$ for some invertible matrix $P$ with coefficients in $\mathcal{F}$. Moreover, there is an $F$-algebra homomorphism of coordinate rings

$$F[XP, \det(XP)^{-1}] \rightarrow F[X_H P, \det(X_H P)^{-1}].$$

Since $X_H P$ is a generic point for an $H$-torsor we have that $XP$ is a generic point for an $\text{SO}_n$-torsor, and therefore the (twisted) Lie algebra associated to the $H$-torsor is contained in that for the $\text{SO}_n$-torsor. In turn, this implies that the generic point $Y$ satisfies an equation with matrix $B = A(f_{ij})$ for some $f_{ij} \in F$. 

Likewise, a specialization $A(f_{ij})$ of $A(Y_{ij})$ with $f_{ij} \in F$, gives a derivation on the coordinate ring $F[XP,\det(XP)^{-1}]$ of an $\text{SO}_n$-torsor. When extended to the quotient field this derivation may have new constants. We get a PVE of $F$ by taking the quotient field of the factor ring

$$F[XP,\det(XP)^{-1}]/M,$$

where $M$ is a maximal differential ideal. The differential Galois group in this case is the closed subgroup of $\text{SO}_n$ consisting of those elements that stabilize $M$.

Finally, it is clear that a fundamental matrix for the equation $\eta' = \eta A(Y_{ij})$ specializes to one for $\eta' = \eta A(f_{ij})$ since, on the one hand, a solution of $\eta' = \eta A(Y_{ij})$ is given by a generic point $XP$ of the $\text{SO}_n$-torsor corresponding to the quadratic form

$$Q = \begin{pmatrix} Y_{11} & Y_{22} & \cdots & 1/Y_{11} \cdots Y_{n-1,n-1} \\ \end{pmatrix}$$

with

$$P = \begin{pmatrix} \sqrt{Y_{11}} \\ \sqrt{Y_{22}} \\ \vdots \\ \sqrt{1/Y_{11} \cdots Y_{n-1,n-1}} \\ \end{pmatrix}$$

and $X$ a generic point of $\text{SO}_n$.

On the other hand, a solution of $\eta' = \eta A(f_{ij})$ is given by a generic point $XP(f_{ij})$ of the $\text{SO}_n$-torsor corresponding to the quadratic form

$$Q(f_{ij}) = \begin{pmatrix} f_{11} \\ f_{22} \\ \vdots \\ 1/f_{11} \cdots f_{n-1,n-1} \\ \end{pmatrix}$$

with

$$P(f_{ij}) = \begin{pmatrix} \sqrt{f_{11}} \\ \sqrt{f_{22}} \\ \vdots \\ \sqrt{1/f_{11} \cdots f_{n-1,n-1}} \\ \end{pmatrix}.$$  

**Note.** In the case of $\text{SO}_3$ we can exhibit a generic point using the classical Euler parametrization:

$$X = \frac{1}{x^2 + y^2 + z^2 + w^2} \begin{pmatrix} x^2 + y^2 - z^2 - w^2 \\ 2yw - 2xz \\ 2xz + 2yw \\ 2y^2 + 2yz \\ x^2 - y^2 + z^2 - w^2 \\ 2xw + 2yz \\ x^2 - y^2 - z^2 + w^2 \\ 2zw - 2xy \\ 2yw + 2xz \\ 2zw + 2zx \\ 2y^2 + 2yz \\ x^2 - y^2 - z^2 + w^2 \\ \end{pmatrix}.$$
obtained by interpreting the quaternion $x + yi + zj + wk$ as an isometry by conjugation on the quadratic space with basis $i, j, k$, where $x, y, z$ and $w$ are indeterminates [2, Theorem 3, Chapter 3]. A generic point for the torsor, is then

$$Y = XP = \frac{1}{x^2 + y^2 + z^2 + w^2} \times$$

$$\begin{pmatrix}
(x^2 + y^2 - z^2 - w^2)\sqrt{a} & 2(xw + yz)\sqrt{b} & 2(yw - xz)\sqrt{ab} \\
2(yz - xw)\sqrt{a} & (x^2 - y^2 + z^2 - w^2)\sqrt{b} & 2(xy + zw)\sqrt{ab} \\
2(xz + yw)\sqrt{a} & 2(zw - xy)\sqrt{b} & (x^2 - y^2 - z^2 + w^2)\sqrt{ab}
\end{pmatrix}.$$  

Clearly, this matrix permits specialization of $a$ and $b$ to any non-zero values.  

\[\square\]

**Remark.** Observe that when the $f_{ii}$ are all 1, the matrix $A(f_{ij})$ then has the form

$$\begin{pmatrix}
0 & f_{12} & f_{13} & \ldots & f_{1n} \\
-f_{12} & 0 & f_{23} & \ldots & f_{2n} \\
-f_{13} & -f_{23} & 0 & \ldots & b_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_{1n} & f_{2n} & f_{3n} & \ldots & 0
\end{pmatrix} \in \text{Lie}(SO_n).$$

Therefore this situation corresponds to the trivial torsor case. In general, if the $f_{ii}$ are (not all equal) constants, the torsor associated to the quadratic form will still be trivial and the specialized matrix will be in a Lie algebra isomorphic to $\text{Lie}(SO_n)$.

5. **Remarks on the general case**

In general, when the matrices $P$ parametrizing the $G$-torsors are not known, it will not be possible to carry out the same kind of explicit construction done here for $SO_n$. In such a situation we can use the generic extension relative to the trivial torsor [6, Definition 3.1, Theorem 3.3] and obtain the extensions corresponding to nontrivial $G$-torsors indirectly:

Assume that $G$ is connected and let $E \supset F$ be a generic extension for $G$ relative to the trivial $G$-torsor, with equation $Z' = A(Y)Z$.

**Theorem 2.** Let $F$ be a differential field with field of constants $C$. There is a PVE $E \supset F$ with differential Galois group $H \leq G$ if and only if there are a finite extension $k \supset F$, a matrix $P$ with coefficients in $k$ and a specialization $Y_i \mapsto f_i \in k$, such that the equation $Z' = Z(P^{-1}A(f_i)P + P^{-1}P')$ gives rise to the extension $E \supset F$.

**Proof.** As before, we let $X$ denote a generic point for $G$ and write $Y = XP$ for a generic point of the $G$-torsor with $E = F(Y)$. The proof then follows from the description of the twisted Lie algebras [7, Section 3] and the Trivialization Lemma shown in Section 3 of this paper. \[\square\]
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References


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