Show all of your work. Circle your answers.
No calculators. Turn off all electronic devices.

Name  ____________________

1. (10 pts) Suppose \( z = f(x, y) = (x + \ln(y))^2 \) and \( x = u^2 - v^2, \ y = e^{uv} \).
   (a) Use the chain rule to compute the partial derivatives. Express your answer in terms of \( u \) and \( v \) and simplify.
   \[
   \frac{\partial z}{\partial u} = \frac{\partial}{\partial x} (x + \ln(y))^2 \cdot (2u) + \frac{\partial}{\partial y} (x + \ln(y))^2 \cdot \frac{1}{y} \cdot y = 4u(x + \ln(y))(-1 + 1) = 0
   \]

   \[
   \frac{\partial z}{\partial v} = \frac{\partial}{\partial x} (x + \ln(y))^2 \cdot (2v) + \frac{\partial}{\partial y} (x + \ln(y))^2 \cdot \frac{1}{y} \cdot y = 4V(v^2 - u^2 - \ln(e^{uv})) = 4V(v^2 - u^2 + lne^{uv}) = 4V(\sqrt{v^2 + 4}^3)
   \]

   \[
   \frac{\partial z}{\partial u} = 0, \quad \frac{\partial z}{\partial v} = 4\sqrt{V^3}
   \]

2. (10 pts) Write an equation for a plane that is tangent to the following surface: \( z - 5 = 2(x - 3) + 8(y - 3) \).
   (a) \( z = x^2 + 4(y - 2)^2 \) at the point \( P_0(1, 3, 5) \).
      \[
      \vec{N} = \langle f_x, f_y, f_z \rangle \bigg|_{P_0} = \langle 2y, 8(y - 2), -1 \rangle = \langle 2, 8, -1 \rangle \quad \Rightarrow \quad 2x + 8y - z - 21 = 0
      \]

      \[
      0 = 2 + 24 - 5 + D = 0 \Rightarrow D = -20
      \]

   (b) \( x^2 + 2xy + z^2 = 4 \) at the point \( P_0(1, 1, 1) \).
      \[
      \vec{N} = \langle f_x, f_y, f_z \rangle \bigg|_{P_0} = \langle 2x + 2y, 2x, 2z \rangle \bigg|_{P_0} = \langle 4, 2, 2 \rangle \quad \Rightarrow \quad 2x + y + z - 4 = 0
      \]

      \[
      0 = 4 + 2 + 2 + D = 0 \Rightarrow D = -8 \Rightarrow 4x + 2y + 2z - 8 = 0
      \]

3. (5 pts) Given the function \( z = f(x, y) = 3x^2y + e^{2y} - 5 \), compute the following:
   (a) Differential of \( f \), \( df = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy \).
   (b) Approximate the increment \( \Delta f(1.99, 0.01) \) using the differential of \( f \).
      \[
      \Delta f \approx df = ((2)(0)(-0.01) + (3, 4 + 2)(0.01)) = (0.14)
      \]
      \[
      \frac{\partial z}{\partial x} = \Delta x = -0.01, \quad \frac{\partial z}{\partial y} = \Delta y = 0.01
      \]

      \[
      f(1.99, 0.01) \approx f(2, 0) + df
      \]
4. (10 pts) Given the function \( z = f(x, y) = x^2y^2 - \sin(x - 1) \), compute the following:

(a) Gradient of \( f \), \( \nabla f(x, y) \) at \( P_0(1, 1) \).
\[
\nabla f\bigg|_{P_0} = \left( \frac{\partial f}{\partial x} \right) \mathbf{i} + \left( \frac{\partial f}{\partial y} \right) \mathbf{j} = (2x^2y - \cos(x-1))\mathbf{i} + 2xy^2\mathbf{j}
\]
\[
\bigg|_{P_0} = (2-1)\mathbf{i} + 2\mathbf{j} = \mathbf{\hat{e}} + 2\mathbf{j}
\]

(b) Directional derivative \( D_uf(x, y) \) at the point \( P_0(1, 1, 1) \) in the direction of \( \mathbf{u} = \mathbf{i} - 3\mathbf{j} \).
\[
\mathbf{u} = \mathbf{i} - 3\mathbf{j}
\]
\[
D_uf(x, y) = \nabla f(1, 1) \cdot \mathbf{u} = \left< 2, 2 \right> \cdot \left< 1, -3 \right> = \frac{2}{10} - \frac{6}{10} = -\frac{4}{10} = -\frac{2}{5}
\]

(c) Direction \( \mathbf{u} \) from \( P_0(1, 1, 1) \) in which \( f \) increases most rapidly and the magnitude of the greatest rate of increase.
\[
\mathbf{u} = \frac{\nabla f}{\|\nabla f\|}
\]
\[
\|\nabla f\| = \sqrt{4 + 4} = \sqrt{8}
\]
\[
D_uf = \|\nabla f\| \cos \theta
\]

5. (15 pts) Consider the function \( z = f(x, y) = x^3 + y^3 - 3xy \).

(a) Find all critical points.

(b) Apply the second derivative test for a function of two variables to classify each critical point as a relative minimum, relative maximum, or saddle point.

\[
f_x = 3x^2 - 3 y = 0 \Rightarrow y = x^2 \quad f_y = 3y^2 - 3x = 0 \Rightarrow x = y^2
\]
\[
y = 0, y = x^2, x = y^2
\]
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 6x^2 - 3y = 0 \Rightarrow y = 2x^2 \\
\frac{\partial f}{\partial y} &= 3y^2 - 3x = 0 \Rightarrow x = y^2
\end{align*}
\]
\[
y = 0, y = x^2, x = y^2
\]
\[
\begin{align*}
D(x, y) &= f_{xx}f_{yy} - (f_{xy})^2 = (6x)(6y) - (-3)^2 \\
D(0, 0) &= -9 < 0 \Rightarrow (0, 0) \text{ saddle point} \\
D(1, 1) &= 36 - 9 > 0 \text{ and } f_{xx}(1, 1) = 6 > 0 \Rightarrow (1, 1) \text{ relative minimum}
\end{align*}
\]

See graph of \( z = f(x, y) \)
plot3d(x^3 + y^3 - 3*x*y, x=-1..2, y=-1..2);