Forchheimer Equations in Porous Media - Part II

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Darcy’s Law:
\[ \alpha u = -\Pi \nabla p, \]

the “two term” law
\[ \alpha u + \beta |u| u = -\nabla p, \]

the “power” law
\[ c^n |u|^{n-1} u + au = -\nabla p, \]

the “three term” law
\[ Au + B |u| u + C |u|^2 u = -\nabla p. \]

Here \( \alpha, \beta, c, A, B, \) and \( C \) are empirical positive constants.
General Forchheimer equations

Generalizing the above equations as follows

\[ g(x, |u|)u = -\nabla p. \]

Let \( G(s) = sg(s) \). Then \( G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|) \). Hence

\[ u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} = -K(|\nabla p|)\nabla p, \]

where

\[ K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi. \]

We derived non-linear Darcy equations from Forchheimer equations.
Equations of Fluids

Let $\rho$ be the density. Continuity equation

$$\frac{d\rho}{dt} = -\nabla \cdot (\rho u),$$

For slightly compressible fluid it takes

$$\frac{d\rho}{dp} = \frac{1}{\kappa} \rho,$$

where $\kappa \gg 1$. Substituting this into the continuity equation yields

$$\frac{d\rho}{dp} \frac{dp}{dt} = -\rho \nabla \cdot u - \frac{d\rho}{dp} u \cdot \nabla p,$$

$$\frac{dp}{dt} = -\kappa \nabla \cdot u - u \cdot \nabla p.$$

Since $\kappa \gg 1$, we neglect the second term in continuity equation

$$\frac{dp}{dt} = -\kappa \nabla \cdot u.$$
Combining the equation of pressure and the Forchheimer equation, one gets after scaling:

\[ \frac{dp}{dt} = \nabla \cdot (K(\nabla p) \nabla p). \]
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\]

Consider the equation on a bounded domain \( U \) in \( \mathbb{R}^3 \). The boundary of \( U \) consists of two connected components: exterior boundary \( \Gamma_e \) and interior (accessible) boundary \( \Gamma_i \).

- On \( \Gamma_e \):

\[
\mathbf{u} \cdot \mathbf{N} = 0 \iff \frac{\partial p}{\partial N} = 0.
\]
Combining the equation of pressure and the Forchheimer equation, one gets after scaling:

\[
\frac{dp}{dt} = \nabla \cdot (K(|\nabla p|) \nabla p).
\]

Consider the equation on a bounded domain \( U \) in \( \mathbb{R}^3 \). The boundary of \( U \) consists of two connected components: exterior boundary \( \Gamma_e \) and interior (accessible) boundary \( \Gamma_i \).

- **On \( \Gamma_e \):**
  
  \[ u \cdot N = 0 \iff \frac{\partial p}{\partial N} = 0. \]

- **Dirichlet condition on \( \Gamma_i \):** \( p(x, t) = \phi(x, t) \) which is known for \( x \in \Gamma_i \).

- **Total flux condition on \( \Gamma_i \):**
  
  \[
  \int_{\Gamma_i} u \cdot N d\sigma = Q(t) \iff \int_{\Gamma_i} K(|\nabla p|) \nabla p \cdot N d\sigma = -Q(t),
  \]
  
  where \( Q(t) \) is known.
We introduce a class of “generalized polynomials with positive coefficients”

**Definition**

A function \( g(s) \) is said to be of class (APPC) if

\[
g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_k s^{\alpha_k} = \sum_{j=0}^{k} a_j s^{\alpha_j},
\]

where \( k \geq 0, 0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k \), and \( a_0, a_1, \ldots, a_k \) are positive coefficients. Notation \( \alpha_k = \text{deg}(g) \).
Lemma

Let $g(s)$ be a function of class (GPPC). Then the function $K$ exists

Moreover, for any $\xi \geq 0$, one has

$$\frac{C_1}{(1 + \xi)^a} \leq K(\xi) \leq \frac{C_2}{(1 + \xi)^a},$$

$$-\theta \frac{K(\xi)}{\xi} \leq K'(\xi) \leq 0,$$

$$(K(\xi)\xi^n)' \geq K(\xi)\xi^{n-1} \left(n - \frac{1}{1 + \lambda}\right) \geq 0,$$

where $a = \frac{\alpha_k}{1 + \alpha_k}$, $0 < \theta < 1$, $n \geq 1$, $\lambda = \frac{1}{\alpha_k}$. 
The Monotonicity

Proposition

Let $g$ be a GPPC. Then $g$ satisfies the **Lambda-Condition**:

$$g(s) \geq \lambda sg'(s), \text{ where } \lambda = 1/\deg(g) > 0.$$ 

The function $F(y) = K(|y|)y$ is strictly monotone on bounded sets. More precisely,

$$(F(y) - F(y')) \cdot (y - y') \geq \frac{\lambda}{\lambda + 1} K(\max\{|y|, |y'|\}) |y' - y|^2.$$
On \( \Gamma_i \) the solution has the form:

\[
p(x, t) = \gamma(t) + \varphi(x) \quad \text{on} \quad \Gamma_i, \quad t > 0,
\]

where the function \( \varphi(x) \) is defined for \( x \in \Gamma_i \) and satisfies

\[
\int_{\Gamma_i} \varphi(x) \, d\sigma = 0.
\]
A priori Estimates - I

Theorem

Let \( p(x, t) \) be a solution of IBVP-I(S) with the boundary profile \((\gamma(t), \varphi(x))\). Then one has for all \( t \geq 0 \) that

\[
\int_U K (|\nabla p(x, t)|) |\nabla p(x, t)|^2 \, dx \leq 2 \int_U K (|\nabla p(x, 0)|) |\nabla p(x, 0)|^2 \, dx \\
+ |U| \int_0^t (\gamma'(\tau))^2 \, d\tau,
\]

If, in addition, \( g(s) \) belongs to class (GPPC), then one has

\[
\int_U |\nabla p(x, t)|^{2-a} \, dx \leq C_1 |U| + C_2 \int_U |\nabla p(x, 0)|^{2-a} \, dx \\
+ C_3 |U| \int_0^t (\gamma'(\tau))^2 \, d\tau.
\]
Define $H(x, t)$ by:

$$H(x, t) = \int_0^{|\nabla p(x,t)|^2} K(\sqrt{s})ds,$$

for $(x, t) \in U \times [0, \infty)$.

Multiplying the equation by $\partial p/\partial t$ and integrate over $U$:

$$\int_U \left( \frac{\partial p}{\partial t} \right)^2 dx = - \int_U K(|\nabla p|) \nabla p \cdot \frac{\partial}{\partial t}(\nabla p) dx - Q(t) \cdot \gamma'(t)$$

$$= \frac{1}{2} \frac{\partial}{\partial t} H(x, t) - Q(t) \cdot \gamma'(t).$$

Note

$$Q^2(t) = \left( \int_U \frac{\partial p}{\partial t} dx \right)^2 \leq |U| \int_U \left( \frac{\partial p}{\partial t} \right)^2 dx.$$
Applying Cauchy’s inequality we obtain

\[ \frac{1}{2} \frac{d}{dt} \int_U H(t, x) dx \leq \frac{|U|}{2} \cdot |\gamma'(t)|^2 - \frac{Q^2(t)}{2|U|}, \]

and thus

\[ \int_U H(t, x) dx \leq \int_U H(0, x) dx + |U| \int_0^t |\gamma'(\tau)|^2 d\tau - \frac{1}{|U|} \int_0^t Q^2(\tau) d\tau. \]

Then use relations

\[ K(|\nabla p(x, t)||\nabla p(x, t)|^2 \leq H[p](x, t) \leq 2K(|\nabla p(x, t)||\nabla p(x, t)|^2, \]

\[ K(|\nabla p(x, t)||\nabla p(x, t)|^2 \sim |\nabla p(x, t)|^{2-a}. \]
Theorem

Let $p(x, t)$ be a solution of IBVP-II(S) with total flux $Q(t)$. Assume that $Q(t) \in C^1([0, \infty))$. Then for any $\delta > 0$, one has

$$\int_U |\nabla p(x, t)|^{2-a} \, dx \leq \Lambda^*(t) - C_1 h_2(t) + \delta \int_0^t e^{\delta(t-\tau)} \left( \Lambda^*(\tau) - C_1 h_2(\tau) \right) \, d\tau,$$

for any $t \geq 0$, where

$$\Lambda^*(t) = L_2 + L_0 h_0(t) + L_1 h_1(t) + 2h_0(t)h_1(t)$$

$$+ C_1 |Q(t)|^{\frac{2-a}{1-a}} + C_\delta \int_0^t |Q'(\tau)|^{\frac{2-a}{1-a}} \, d\tau,$$

with the positive numbers $L^*, L_0, L_1$ depending on the initial data; while $C, C_\delta$ independent of the initial data;
Theorem

and the functions $h_i(t)$, $i = 0, 1, 2$ are defined by

\[
h_0(t) = \int_0^t |Q(\tau)| d\tau, \quad h_1(t) = \int_0^t |Q'(\tau)| d\tau.
\]
Theorem

Let $g(s)$ be of class (GPPC). Let $p_\gamma(x, t)$ be a solution of IBVP-I(S) with known total flux $Q(t)$ and known boundary profile $(\gamma(t), \varphi(x))$. Let $p(x, t)$ be a solution of IBVP-II(S) with total flux $Q(t)$ and boundary profile $(B(t), \varphi(x))$, where $B(t)$ is not given but bounded from above. Suppose that $Q \in C^1([0, \infty))$, $Q'(t) \geq 0$ and $B(t) \leq B_0 < \infty$. Then

$$\int_U |\nabla p(x, t)|^{2-a} dx \leq C_1 \left( \int_U |\nabla p_\gamma(x, t)|^{2-a} dx + |Q(t)|^{\frac{2-a}{1-a}} \right. + |Q(t)| |\gamma(t)|) + L_0,$$

the positive number $L_0$ depends on the initial data of $p_\gamma$ and $p$; and $C_1$, $C_2$ are positive constants.
A priori Estimates - III

**Theorem**

Consequently,

\[
\int_{U} \left| \nabla p(x, t) \right|^{2-a} dx \leq \int_{0}^{t} \left| \gamma'(\tau) \right|^{2} d\tau + \left| Q(t) \right|^{\frac{2-a}{1-a}} + \left| Q(t) \right| \left| \gamma(t) \right| + L_0.
\]
Lemma

Let the function $g$ be of the class (GPPC). For any functions $f$, $p_1$ and $p_2$, and for $1 \leq q < 2$, one has

$$
\left( \int_{U} |f|^q dx \right)^{2/q} \leq C \left( \int_{U_1} K(|\nabla p_1|) |f|^2 dx + \int_{U_2} K(|\nabla p_2|) |f|^2 dx \right)
\times \left\{ 1 + \max \left( \| \nabla p_1 \|_{L^{2-q}(U)}^{aq}, \| \nabla p_2 \|_{L^{2-q}(U)}^{aq} \right) \right\}^a,
$$

where

$$
U_1 = \{ x : |\nabla p_1(x)| \geq |\nabla p_2(x)| \}, \quad U_2 = \{ x : |\nabla p_1(x)| < |\nabla p_2(x)| \}.
$$
Lemma

Consequently

\[
\int_U \left( K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2 \right) \cdot (\nabla p_1 - \nabla p_2) \, dx \\
\geq C \left( \int_U |\nabla (p_1 - p_2)|^q \, dx \right)^{2/q} \\
\times \left\{ 1 + \max \left( \|\nabla p_1\|_{L^{\frac{aq}{2-q}}(U)}, \|\nabla p_2\|_{L^{\frac{aq}{2-q}}(U)} \right) \right\}^{-a}.
\]
Theorem

Assume that $\deg(g) \leq \frac{4}{d-2}$. Suppose $p_1(x, t), p_2(x, t)$ are two solutions of IBVP-I(S) with the same boundary profile $(\gamma(t), \varphi(x))$. Then

$$\|p_1(\cdot, t) - p_2(\cdot, t)\|_{L^2(U)} \leq \|p_1(\cdot, 0) - p_2(\cdot, 0)\|_{L^2(U)} \cdot \exp\left[-C \int_0^t \Lambda^{-b}(\tau) d\tau\right],$$

for all $t \geq 0$, where

$$\Lambda(t) = 1 + \int_U |\nabla p_1(x, 0)|^{2-a} dx + \int_U |\nabla p_2(x, 0)|^{2-a} dx + \int_0^t |\gamma'(\tau)|^2 d\tau.$$
Corollary

If \( \int_0^t \left| \gamma'(\tau) \right|^2 d\tau = O(t^r) \) as \( t \to \infty \), for some \( 0 < r < 1/b \), then

\[
\|z(\cdot, t)\|_{L^2(U)} \leq C_1 e^{-C_2 t^\varepsilon} \|z(\cdot, 0)\|_{L^2(U)},
\]

where \( \varepsilon = 1 - rb > 0 \).

Example

Suppose \( \gamma(t) = a_0 + a_1 t^\beta \), where \( a_1 \neq 0 \), for all \( t > T \), where \( T > 0 \). Then

(i) if \( \beta < 1/a \) then (12) holds for \( \varepsilon = (1 - 2\beta)b + 1 \);

(ii) if \( \beta = 1/a \) then \( \|z(\cdot, t)\|_{L^2} \leq C(1 + t)^{-s}\|z(\cdot, 0)\|_{L^2} \) for some number \( s > 0 \).
Solutions $p_k$ with Dirichlet data $(\gamma_k(t), \varphi(x))$ and flux $Q_k(t)$, for $k = 1, 2$. We assume that for $k = 1, 2$:

$$\int_0^t |\gamma_k'(\tau)|^2 d\tau \leq \lambda_0(t),$$

where $\lambda_0 \in C([0, \infty))$. Let

$$\Lambda_0(t) = (1 + \lambda_0(t))^b.$$

where $b = a/(2 - a)$,

$$Z(t) = \int_U z^2(x, t)dx, \quad F_1(t) = e^{-C_0 A_1 \int_0^t \Lambda_0^{-1}(\tau)d\tau},$$

where $C_0 > 0$ is a constant independent of the solutions.
Theorem

Assume \( \text{deg}(g) \leq \frac{4}{d-2} \). Let \( \overline{p}_k(x, t) = p_k(x, t) - \gamma_k(t) \) for \( k = 1, 2 \). Let

\[
\overline{z}(x, t) = \overline{p}_1(x, t) - \overline{p}_2(x, t), \quad \text{and} \quad \overline{Z}(t) = \int_U \overline{z}^2(x, t)dx.
\]

Then one has for all \( t \geq 0 \) that

\[
\overline{Z}(t) \leq F_1(t)\overline{Z}(0) + C_1A_1^{-1}F_1(t) \int_0^t \Lambda_0(\tau)(\Delta'_\gamma(\tau))^2F_1^{-1}(\tau)d\tau.
\]

Consequently,

\[
Z(t) \leq 2F_1(t)\overline{Z}(0)+2C_2A_1^{-1}F_1(t) \int_0^t \Lambda_0(\tau)(\Delta'_\gamma(\tau))^2F_1^{-1}(\tau)d\tau+2|\Delta_\gamma(t)|^2.
\]
Corollary

Assume that

\[
\int_0^\infty \Lambda_0^{-1}(\tau) d\tau = \infty, \\
\int_0^\infty \Lambda_0(\tau) (\Delta_\gamma'(\tau))^2 F_1^{-1}(\tau) d\tau = \infty, \\
\lim_{t \to \infty} \Lambda_0(t) \Delta_\gamma'(t) = \lambda_1 \in \mathbb{R}.
\]

Then

\[
\overline{Z}(t) \leq F_1(t) \overline{Z}(0) + C_1 C_0^{-1} A_1^{-2} \lambda_1^2 + \epsilon(t),
\]

where \( \epsilon(t) \to 0 \) as \( t \to \infty \).

Consequently, if \( \lambda_1 = 0 \) then

\[
\lim_{t \to \infty} \overline{Z}(t) = 0.
\]
Example

Suppose $\gamma_k(t) = a_{0,k} + a_{1,k} t^{\beta_k}$, where $a_{i,k} \neq 0$, for $i = 0, 1$, and $k = 1, 2$. Let $\beta = \max\{\beta_1, \beta_2\}$ and $\alpha = \deg(g)$. If $\beta < 2(\alpha + 1)/(3\alpha + 2)$ then $\lim_{t \to \infty} \overline{Z}(t) = 0$.

Example

Suppose $\gamma_1(t) = a_{0,1} + a_{1,1} t^{\beta}$ and $\gamma_2(t) = \gamma_1(t) + \Delta_\gamma(t)$, where $\Delta_\gamma(t) = a_{0,3} + a_{1,3} t^r$, with $r < \beta$ and $a_{1,3} \neq 0$. Then $\lim_{t \to \infty} \overline{Z}(t) = 0$ if $\beta < 1/a$ and $r < 1 - (2\beta - 1)b$. 

Let $p_1$ and $p_2$ be two solutions of IBVP-II(S). Let $\delta > 0$ be fixed, and let

$$\Lambda_k^*(t) = \left(1 + \int_0^t |Q_k(\tau)| d\tau\right) \left(1 + \int_0^t |Q'_k(\tau)| d\tau\right) + |Q(t)|^{\frac{2-a}{1-a}} + \int_0^t |Q'(\tau)|^{\frac{2-a}{1-a}} d\tau.$$ 

We assume that

$$e^{-\delta t} \Lambda_k^*(t) + \delta \int_0^t e^{-\delta \tau} \Lambda_k^*(\tau) d\tau \leq \tilde{\lambda}_0(t), \quad t \geq 0, \quad k = 1, 2,$$

where the function $\tilde{\lambda}_0(t)$ is known and belongs to $C([0, \infty))$. 
Theorem

Given $\delta$, $T$ and a solution $p_1(x,t)$ with $Q_1, Q'_1 \in L^\infty_{\text{loc}}([0, \infty))$. For any $\varepsilon > 0$, there is $\sigma > 0$ depending on $\delta$, $T$, $\|Q_1\|_{L^\infty(0,T)}$, $\|Q'_1\|_{L^\infty(0,T)}$ and the initial data of $p_1$, such that if

$$\int_U |z(x,0)|^2 \, dx, \int_U |\nabla z(x,0)|^{2-a} \, dx, \|\Delta Q\|_{L^\infty(0,T)}, \|\Delta' Q\|_{L^\infty(0,T)} < \sigma,$$

then

$$\int_U |z(x,t)|^2 \, dx < \varepsilon, \quad \text{for all} \quad t \in [0, T].$$
More specifically, there is $L > 0$ depending on $\delta$, $T$, $\|Q_1\|_{L^\infty(0,T)}$, $\|Q_1'\|_{L^\infty(0,T)}$ and the initial data of $p_1$, such that

$$\sup_{t \in [0,T]} \int_{U} |p_1(x, t) - p_2(x, t)|^2 dx \leq L \left( \int_{U} |p_1(x, 0) - p_2(x, 0)|^2 dx + \left( \sup_{t \in [0,T]} |\Delta Q(t)| \right)^2 \right).$$
Let \( p_1 \) and \( p_2 \) be two solutions of IBVP-II(S) satisfying condition

\[
\int_0^t |\gamma'_k(\tau)|^2 d\tau \leq \lambda_0(t).
\]

Let \( \Lambda_0(t) = (1 + \lambda_0(t))^b \). Assume \( \deg(g) \leq \frac{4}{d-2} \). Let

\[
\bar{p}_k(x, t) = p_k(x, t) + |U|^{-1} \int_0^t Q_k(\tau) d\tau \quad \text{for} \quad k = 1, 2,
\]

\[
\bar{z}(x, t) = \bar{p}_1(x, t) - \bar{p}_2(x, t) - |U|^{-1} \int_U (p_1(x, 0) - p_2(x, 0))dx,
\]

and

\[
\bar{Z}(t) = \int_U \bar{z}^2(x, t)dx.
\]
Theorem

Then one has for all $t \geq 0$ that

$$Z(t) \leq F_1(t)Z(0) + C_2 A_1^{-1} F_1(t) \int_0^t \Lambda_0(\tau)(\Delta_Q(\tau))^2 F^{-1}_1(\tau) d\tau.$$
Define

\[ I_Q(t) = \left( \int_0^t \Delta Q(\tau) d\tau \right)^2 \quad \text{and} \quad I_z(t) = \left( \int_U z(x, t) dx \right)^2. \]

**Theorem**

Assume \( \deg(g) < \frac{4}{d-2} \). One has for all \( t \geq 0 \) that

\[
Z(t) \leq F_1(t) \left[ Z(0) + C_1 A_1 \int_0^t \frac{I_Q(\tau)}{F_1(\tau) \Lambda_0(\tau)} d\tau \\
+ C_2 A_1 I_z(0) \int_0^t \frac{1}{F_1(\tau) \Lambda_0(\tau)} d\tau + C_3 A_1^{-1} \int_0^t \frac{\Lambda_0(\tau) \Delta^2 Q(\tau)}{F_1(\tau)} d\tau \right].
\]
Theorem

Let \( p_1 \) and \( p_2 \) be two solutions to IBVP-I(S). Assume that

\[
\Delta^2_Q(t) \leq q_0 I_Q(t) + q_1 \Delta^2_Q(0) + q_2, \quad \text{some} \quad q_0, q_1, q_2 \geq 0.
\]

Then one has

\[
Z(t) \leq F_1(t) \left[ Z(0) + C_1(q_1 \Delta_Q(0) + q_2 + I_z(0)) \int_0^t \frac{1}{F_1(\tau) \Lambda_0(\tau)} \, d\tau 
+ C_2 \int_0^t \frac{\Lambda_0(\tau) \Delta^2_\gamma(\tau)}{F_1(\tau)} \, d\tau \right].
\]
Remark

From physical point of view, the above condition restricts the amplitude of possible spikes of the total flux from too large deviation, and this, in fact, is not stringent. Indeed, the condition implies

\[ |\Delta Q(t)| \leq C_1 \int_0^t |\Delta Q(\tau)| d\tau + C_2, \]

hence by Gronwall’s inequality: \( |\Delta Q(t)| \leq C_3 e^{C_1 t} + C_4 \). It means that \( |\Delta Q(t)| \) cannot grow faster than exponential functions.