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Introduction

- Darcy’s Law:

\[ \alpha u = -\Pi \nabla p, \]

Here \( \alpha, \beta, c, A, B, \) and \( C \) are empirical positive constants.

Above \( B = B(x) \) is positive definite, \( \Pi = \Pi(x) \) is the (normalized) permeability tensor, positive definite, symmetric, and satisfies

\[ k_1 \geq y^\top \Pi y / |y|^2 \geq k_0 > 0, \]

the norms are \( |y| = \left( \sum_{i=1}^d y_i^2 \right)^{1/2} \) and \( |u|^B = \sqrt{(u^\top Bu)} \).
Introduction

- **Darcy’s Law:**
  \[ \alpha u = -\Pi \nabla p, \]

- the “two term” law
  \[ \alpha u + \beta |u|_B u = -\Pi \nabla p, \]

- the “power” law
  \[ c^n |u|^{n-1}_B u + au = -\Pi \nabla p, \]

- the “three term” law
  \[ Au + B |u|_B u + C |u|_B^2 u = -\Pi \nabla p. \]

Here \( \alpha, \beta, c, A, B, \) and \( C \) are empirical positive constants. Above \( B = B(x) \) is positive definite, \( \Pi = \Pi(x) \) is the (normalized) permeability tensor, positive definite, symmetric, and satisfies

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General Forchheimer equations

Generalizing the above equations as follows

$$g(x, |u|_B)u = -\Pi \nabla p.$$ 

Let $B = I_3$, $|u|_B = |u|$, $g = g(|u|)$, and $\Pi = I_3$. Solve for $u$ in terms of $\nabla p$.

Let $G(s) = sg(s)$. Then $G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|)$. Hence

$$u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} = -K(|\nabla p|) \nabla p,$$

where

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

We derived non-linear Darcy equations from Forchheimer equations.
Let $\rho$ be the density. Continuity equation

$$\frac{d\rho}{dt} = -\nabla \cdot (\rho u),$$

For slightly compressible fluid it takes

$$\frac{d\rho}{dp} = \frac{1}{\kappa} \rho,$$

where $\kappa \gg 1$. Substituting this into the continuity equation yields

$$\frac{d\rho}{dp} \frac{dp}{dt} = -\rho \nabla \cdot u - \frac{d\rho}{dp} u \cdot \nabla p,$$

$$\frac{dp}{dt} = -\kappa \nabla \cdot u - u \cdot \nabla p.$$

Since $\kappa \gg 1$, we neglect the second term in continuity equation

$$\frac{dp}{dt} = -\kappa \nabla \cdot u.$$
Combining the equation of pressure and the Forchheimer equation, one gets after scaling:

\[
\frac{dp}{dt} = \nabla \cdot \left( K(\nabla p) \nabla p \right).
\]
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\[ \frac{dp}{dt} = \nabla \cdot (K(\nabla p)|\nabla p) \cdot . \]

Consider the equation on a bounded domain \( U \) in \( \mathbb{R}^3 \). The boundary of \( U \) consists of two connected components: exterior boundary \( \Gamma_e \) and interior (accessible) boundary \( \Gamma_i \).

- On \( \Gamma_e \):
  \[ u \cdot N = 0 \iff \frac{\partial p}{\partial N} = 0. \]
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Consider the equation on a bounded domain \( U \) in \( \mathbb{R}^3 \). The boundary of \( U \) consists of two connected components: exterior boundary \( \Gamma_e \) and interior (accessible) boundary \( \Gamma_i \).

- On \( \Gamma_e \):
  \[
  u \cdot N = 0 \iff \frac{\partial p}{\partial N} = 0.
  \]

- Dirichlet condition on \( \Gamma_i \): \( p(x, t) = \phi(x, t) \) which is known for \( x \in \Gamma_i \).

- Total flux condition on \( \Gamma_i \):
  \[
  \int_{\Gamma_i} u \cdot N d\sigma = Q(t) \iff \int_{\Gamma_i} K(\nabla p) \nabla p \cdot N d\sigma = -Q(t),
  \]
  where \( Q(t) \) is known.
Let $F$ be a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$.

- $F$ is (positively) monotone if
  \[(F(y') - F(y)) \cdot (y' - y) \geq 0, \text{ for all } y', y \in \mathbb{R}^d.\]

- $F$ is strictly monotone if there is $c > 0$ such that
  \[(F(y') - F(y)) \cdot (y' - y) \geq c|y' - y|^2, \text{ for all } y', y \in \mathbb{R}^d.\]

- $F$ is strictly monotone on bounded sets if for any $R > 0$, there is a positive number $c_R > 0$ such that
  \[(F(y') - F(y)) \cdot (y' - y) \geq c_R|y' - y|^2, \text{ for all } |y'| \leq R, |y| \leq R.\]
Existence of function $K(\cdot)$

Aim $K_g(\xi) = 1/g(G^{-1}(\xi))$ exists.

**G-Conditions:**

$$g(0) > 0, \quad \text{and} \quad g'(s) \geq 0 \text{ for all } s \geq 0.$$ 

**Lemma**

Let $g(s)$ satisfy the G-Conditions. Then $K(\xi)$ exists. Moreover, for any $\xi \geq 0$, one has

$$K'(\xi) = -K(\xi) \frac{g'(s)}{\xi g'(s) + g^2(s)} \leq 0,$$

$$(K(\xi)\xi^n)' = K(\xi)\xi^{n-1} \left(n - \frac{\xi g'(s)}{\xi g'(s) + g^2(s)}\right) \geq 0,$$

for any $n \geq 1$, where $s = G^{-1}(\xi)$. 

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Forchheimer Equations - I

The Monotonicity

Aim: $F(y) = K(|y|)y$ is monotone.

**Lambda-Condition:**

$$g(s) \geq \lambda s g'(s), \text{ some } \lambda > 0.$$ 

**Proposition**

Let $g(s)$ satisfy the G-Conditions. Then $F(y) = K(|y|)y$ is monotone.
The Monotonicity

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Lambda-Condition:

\[
g(s) \geq \lambda s g'(s), \quad \text{some } \lambda > 0.
\]

**Proposition**

Let \( g(s) \) satisfy the G-Conditions. Then \( F(y) = K(|y|)y \) is monotone. In addition, if \( g \) satisfies the Lambda-Condition, then \( F(y) \) is strictly monotone on bounded sets. More precisely,

\[
(F(y) - F(y')) \cdot (y - y') \geq \frac{\lambda}{\lambda + 1} K(\max\{|y|, |y'|\}) |y' - y|^2.
\]

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Forchheimer Equations - I

We introduce a class of “algebraic polynomials with positive coefficients”

Definition
A function \( g(s) \) is said to be of class (APPC) if

\[
g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_k s^{\alpha_k} = \sum_{j=0}^{k} a_j s^{\alpha_j},
\]

where \( k \geq 0 \), \( 0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k \), and \( a_0, a_1, \ldots, a_k \) are positive coefficients.

Proposition
Let \( g(s) \) be a function of class (APPC). Then \( g \) satisfies G-Conditions and Lambda-Condition. Consequently, \( F(y) = K_g(|y|)y \) is strictly monotone on bounded sets.
Proposition

Let \( g(s) \) satisfy the G-Conditions. Let \( p_1 \) and \( p_2 \) are two solutions of IBVP with the same Dirichlet condition on \( \Gamma_i \). Then

\[
\int_U |p_1(x, t) - p_2(x, t)|^2 \, dx \leq \int_U |p_1(x, 0) - p_2(x, 0)|^2 \, dx.
\]

Consequently, if \( p_1(x, 0) = p_2(x, 0) = p_0(x) \in L^2(U) \), then \( p_1(x, t) = p_2(x, t) \) for all \( t \).
Asymptotic Stability for IBVP-I

Proposition

Assume additionally that \( g(s) \) satisfies the Lambda-Condition, and

\[
\nabla p_1, \nabla p_2 \in L^\infty(0, \infty; L^\infty(U)),
\]

then

\[
\int_U |p_1(x, t) - p_2(x, t)|^2 dx \leq e^{-c_1 K(M)t} \int_U |p_1(x, 0) - p_2(x, 0)|^2 dx,
\]

for all \( t \geq 0 \), where

\[
M = \max \{ \| \nabla p_1 \|_{L^\infty(0, T; L^\infty(U))}, \| \nabla p_2 \|_{L^\infty(0, T; L^\infty(U))} \}.
\]

Consequently,

\[
\lim_{t \to \infty} \int_U |p_1(x, t) - p_2(x, t)|^2 dx = 0.
\]
Proposition

Let \( g(s) \) satify the G-Conditions. Let \( p_1 \) and \( p_2 \) are two solutions of IBVP with the same total flux \( Q(t) \) on \( \Gamma_i \).
Assume that \( (p_1 - p_2)|_{\Gamma_i} \) is independent of the spatial variable \( x \).
Then
\[
\int_U |p_1(x, t) - p_2(x, t)|^2 dx \leq \int_U |p_1(x, 0) - p_2(x, 0)|^2 dx.
\]
Consequently, if \( p_1(x, 0) = p_2(x, 0) = p_0(x) \in L^2(U) \), then \( p_1(x, t) = p_2(x, t) \) for all \( t \).
Proposition

Let \( g(s) \) satisfy the G-Conditions. Let \( p_1 \) and \( p_2 \) are two solutions of IBVP with the same total flux \( Q(t) \) on \( \Gamma_i \).
Assume that \( (p_1 - p_2)\rvert_{\Gamma_i} \) is independent of the spatial variable \( x \).
Then
\[
\int_U \left| p_1(x, t) - p_2(x, t) \right|^2 dx \leq \int_U \left| p_1(x, 0) - p_2(x, 0) \right|^2 dx.
\]

Consequently, if \( p_1(x, 0) = p_2(x, 0) = p_0(x) \in L^2(U) \), then \( p_1(x, t) = p_2(x, t) \) for all \( t \).

Similar results to the Dirichlet condition hold when \( g \) satisfies the Lambda-Condition.
Pseudo Steady State Solutions

**Definition**

The solution $\bar{p}(x, t)$ is called a pseudo steady state (PSS) if

$$\frac{\partial \bar{p}(x, t)}{\partial t} = \text{const}, \quad \text{for all} \quad t.$$ 

This leads to the equation

$$\frac{\partial \bar{p}(x, t)}{\partial t} = -A = \nabla \cdot (K(|\nabla \bar{p}|)\nabla \bar{p}),$$

where $A$ is a constant.

Integrating the equation over $U$ gives

$$A|U| = -\int_{\Gamma_i} (K(|\nabla \bar{p}|)\nabla \bar{p}) \cdot Nd\sigma = \int_{\Gamma_i} u \cdot Nd\sigma = Q(t).$$

Therefore, the total flux of a PSS solution is

$$Q(t) = A|U| = Q, \quad \text{for all} \quad t.$$
Write

$$\bar{p}(x, t) = -At + h(x),$$

one has $\nabla p = \nabla h$, hence $h$ and $p$ satisfy the same PDE and boundary condition on $\Gamma_e$. On $\Gamma_i$, in general, we consider

$$h(x) = \varphi(x) \quad \text{on} \quad \Gamma_i.$$

In the case $\varphi(x) = \text{const.}$ on $\Gamma_i$, by shifting has

$$\bar{p}(x, t) = -At + B + W(x),$$

where $A$ and $B$ are two numbers, and $W(x) = W_A(x)$ satisfies

$$-A = \nabla \cdot (K(|\nabla W|)\nabla W),$$

$$\frac{\partial W}{\partial N} = 0 \quad \text{on} \quad \Gamma_e,$$

$$W = 0 \quad \text{on} \quad \Gamma_i,$$
Proposition

Let \( g(s) \) belong to class (APPC). Then for any number \( A \), the basic profile \( W = W_A \) satisfies

\[
\| \nabla W \|_{L^{2-a}} \leq M = C(|A| + 1)^{1/(1-a)},
\]

where \( a = \frac{\alpha_k}{\alpha_k+1} \).
Continuous dependence of PSS Solutions on the total flux

Proposition

Let \( g(s) \) be of class (APPC). Then there exists constant \( C \) such that for any \( A_1, A_2 \), the corresponding profiles \( W_1, W_2 \) satisfy

\[
\left( \int_U |\nabla (W_1 - W_2)|^{2-a} \, dx \right)^{\frac{2}{2-a}} \leq C \, M \, |A_1 - A_2| \int_U |W_1 - W_2| \, dx,
\]

where \( M = (\max(|A_1|, |A_2|) + 1)^{a/(1-a)} \) and \( C > 0 \) is independent of \( A_1 \) and \( A_2 \).
Under the same assumptions as the previous proposition, there exists a constant $C$ such that
\[
\|\nabla (W_1 - W_2)\|_{L^{2-a}} \leq C \cdot M \cdot |A_1 - A_2|,
\]
where $M = \left[\max(|A_1|, |A_2|) + 1\right]^{a/(1-a)}$. 