Chapter 3

The Implicit Function Theorem and Its Applications

We present the Inverse Mapping Theorem first (Theorem 3.18 in the text) and then the Implicit Function Theorem (Theorem 3.9 in the text).

**Theorem 3.1** (The inverse mapping theorem). Let $U$ and $V$ be open sets in $\mathbb{R}^n$ and $a \in U$. Let $f : U \to V$ be a mapping of class $C^1$ and $b = f(a)$. Suppose $Df(a)$ is invertible, that is, $\det Df(a) \neq 0$. Then there exist neighborhoods $M \subset U$ of $a$ and $N \subset V$ of $b$, so that $f$ is a one-to-one map from $M$ onto $N$, and the inverse map $f^{-1}$ from $N$ to $M$ is also of class $C^1$. Moreover, if $x \in M$ and $y = f(x) \in N$, then $D(f^{-1})(y) = [Df(x)]^{-1}$.

Read the example on p. 138 of the textbook.

Let $F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$. For $x \in \mathbb{R}^n, y \in \mathbb{R}^k$, $F(x, y) = (F_1, F_2, \ldots, F_k) \in \mathbb{R}^k$. We use the following notation

$$D_xF = \left( \partial_{x_j} F_i \right)_{1 \leq i \leq k \atop 1 \leq j \leq n} = \begin{pmatrix} \partial_{x_1} F_1 & \partial_{x_2} F_1 & \cdots & \partial_{x_n} F_1 \\ \partial_{x_1} F_2 & \partial_{x_2} F_2 & \cdots & \partial_{x_n} F_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} F_k & \partial_{x_2} F_k & \cdots & \partial_{x_n} F_k \end{pmatrix},$$
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\[
D_y F = \left( \partial_{y_j} F_i \right)_{1 \leq i, j \leq k} = \begin{pmatrix}
\partial_{y_1} F_1 & \partial_{y_2} F_1 & \ldots & \partial_{y_k} F_1 \\
\partial_{y_1} F_2 & \partial_{y_2} F_2 & \ldots & \partial_{y_k} F_2 \\
\vdots & \vdots & \ddots & \vdots \\
\partial_{y_1} F_k & \partial_{y_2} F_k & \ldots & \partial_{y_k} F_k
\end{pmatrix}.
\]

Note that \( D_x F \) is a \( k \times n \) matrix, \( D_y F \) is a \( k \times k \) matrix and the derivative of \( F \) is

\[ DF = \begin{pmatrix}
D_x F & D_y F
\end{pmatrix}, \]

a \( k \times (n + k) \) matrix.

**Theorem 3.2.** Let \( U \subset \mathbb{R}^n \times \mathbb{R}^k \) be open and \( F : U \to \mathbb{R}^k \) is of class \( C^1 \). Let \( a \in \mathbb{R}^n \), \( b \in \mathbb{R}^k \) such that \( (a, b) \in U \). Suppose \( F(a, b) = 0 \) and the matrix \( B = D_y F(a, b) \) is invertible, that is, \( \det B \neq 0 \). Then there are positive numbers \( r_0 \) and \( r_1 \) such that

- (i) For all \( x \in B(r_0, a) \), there exists unique \( y \in B(r_1, b) \) such that \((x, y) \in U \) and \( F(x, y) = 0 \).

We define the function \( f : B(r_0, a) \to B(r_1, b) \) as follows: for each \( x \in B(r_0, a) \), \( f(x) \) is that unique \( y \in B(r_1, b) \).

- (ii) The function \( f \) above is of class \( C^1 \) and \( F(x, f(x)) = 0 \) for all \( x \in B(r_0, a) \). Consequently, for \( x \in B(r_0, a) \) and \( y = f(x) \), we have

\[
Df(x) = -[D_y F(x, y)]^{-1} D_x F(x, y),
\]

whenever \( D_y F(x, y) \) is invertible.

**Example 3.3.** Consider the problem of solving

\[
x - yu^2 = 0, \quad xy + uv = 0
\]

for \( u \) and \( v \) as functions of \( x \) and \( y \).

Let \( n = k = 2 \). Set \( F = (F_1, F_2) = (x - yu^2, xy + uv) \). We have

\[
A = D_{(x,y)} F = \begin{pmatrix}
1 & -u^2 \\
y & x
\end{pmatrix},
\]
\[B = D_{(u,v)}F = \begin{pmatrix} -2yu & 0 \\ v & u \end{pmatrix}.\]

We have \(\det D_{(u,v)}F(x, y, u, v) = -2yu^2\). Therefore, for any solution \((x_0, y_0, u_0, v_0)\) of (3.1) such that \(y_0u_0 \neq 0\), we can solve (3.1) for \((u, v) = f(x, y) = (u(x, y), v(x, y))\) nearby the given point \((x_0, y_0, u_0, v_0)\).

For example, let \((x_0, y_0, u_0, v_0) = (1, 1, 1, -1)\) be a solution of (3.1). We want to find also \(Df(1, 1)\). We have

\[A = D_{(x,y)}F(1,1,1,-1) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\]

\[B = D_{(u,v)}F(1,1,1,-1) = D_{(u,v)}F = \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix}.\]

It is known that if \(ad - bc \neq 0\), the inverse matrix of \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) is

\[\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.\]

We have \(\det B = -2 \neq 0\) and hence \(B^{-1} = \begin{pmatrix} -1/2 & 0 \\ -1/2 & 1 \end{pmatrix}\). Thus

\[Df(1, 1) = -B^{-1}A = -\begin{pmatrix} -1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\]

\[= -\begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & -3/2 \end{pmatrix}.\]

This implies

\[\partial_x u(1, 1) = 1/2, \; \partial_y u(1, 1) = -1/2, \; \partial_x v(1, 1) = -1/2, \; \partial_y v(1, 1) = -3/2.\]