Chapter 1

Setting the stage

1.1 Euclidean spaces and vectors

Let \( n \) be a natural number, i.e. \( n = 1, 2, 3, \ldots \). The \( n \)-dimensional Euclidean space is the set of ordered \( n \)-tuples of real numbers. We denote this space by \( \mathbb{R}^n \). Then

\[
\mathbb{R}^n = \{ x = (x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R} \},
\]

where \( \mathbb{R} \) denotes the set of real numbers. In fact, \( \mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \) the Cartesian product of \( \mathbb{R} \). Each element in \( x = (x_1, x_2, \ldots, x_n) \) is called a vector with components \( x_1, x_2, \ldots, x_n \). Other notations for vectors can be bold letters \( \mathbf{x} \) or underlined letters \( \underline{x} \); however we will not use these in this note.

The zero vector of \( \mathbb{R}^n \) is simply \( 0 = (0, 0, \ldots, 0) \).

Let \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \) be two vectors in \( \mathbb{R}^n \) and \( c \in \mathbb{R} \). We define the following operations:

- **Addition:** \( x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \),
- **Scalar product:** \( cx = (cx_1, cx_2, \ldots, cx_n) \),
- **Dot product:** \( x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \),

The norm (or the length) of \( x \) is

\[
|x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.
\]
Denote \(-x = (-1)x = (-x_1, -x_2, \ldots, -x_n)\).

Some immediate properties:

\[
x + y = y + x, (x+y) + z = x + (y+z), c(x+y) = cx+cy, x + (-x) = 0,
\]

(1.3)

\[
|cx| = |c||x|, |−x| = |x|.
\]

(1.4)

**Proposition 1.1** (Cauchy-Schwarz’s inequality). For any \(a, b \in \mathbb{R}^n\),

\[
|a \cdot b| \leq |a||b|.
\]

(1.5)

**Proof.** See text, p.5. 

\(\square\)

**Example 1.2.** For \(n = 2, a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2\), we have

\[
|a_1b_1 + a_2b_2| \leq \sqrt{a_1^2 + a_2^2} \sqrt{b_1^2 + b_2^2}.
\]

(1.6)

For \(n = 3, a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{R}^3\), we have

\[
|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}.
\]

(1.7)

**Proposition 1.3** (The triangle inequality). For any \(a, b \in \mathbb{R}^n\),

\[
|a + b| \leq |a| + |b|.
\]

(1.8)

Consequently,

\[
|a - b| \geq | |a| - |b| |.
\]

(1.9)

**Corollary 1.4.** For any \(x, y, z \in \mathbb{R}^n\),

\[
|x - y| \leq |x - z| + |z - y|.
\]

(1.10)

\[
|x| \geq | |y| - |x - y| |.
\]

(1.11)

Relation between the norm of \(x\) and that of its components: Let \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) and \(M = \max\{|x_1|, |x_2|, \ldots, |x_n|\};\) then

\[
M \leq |x| \leq \sqrt{n}M.
\]

(1.12)
1.2 Subsets of Euclidean space

Let $a \in \mathbb{R}^n$ and $r > 0$. The (open) ball $B(r, a)$ is the set of all points whose distance to $a$ is less than $r$,

$$B(r, a) = \{x \in \mathbb{R}^n : |x - a| < r\}. \quad (1.13)$$

We can also define the closed ball

$$B'(r, a) = \{x \in \mathbb{R}^n : |x - a| \leq r\}. \quad (1.14)$$

Let $S$ be a subset of $\mathbb{R}^n$. Then the complement of $S$ in $\mathbb{R}^n$ is $S^c$, the set of all points in $\mathbb{R}^n$ that are not in $S$:

$$S^c = \mathbb{R}^n \setminus S = \{x \in \mathbb{R}^n : x \notin S\}. \quad (1.15)$$

**Example 1.5.** If $S = B(r, a)$, then $S^c = \{x \in \mathbb{R}^n : |x - a| \geq r\}$. If $S = B'(r, a)$, then $S^c = \{x \in \mathbb{R}^n : |x - a| > r\}$.

**Definition 1.6.** Let $S$ be a subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$.

$x$ is called an interior point of $S$ if there is $r > 0$ such that $B(r, x) \subset S$.

We denote the set of interior points of $S$ by $S^{\text{int}}$:

$$S^{\text{int}} = \{x \in \mathbb{R}^n : \exists r > 0, B(r, x) \subset S\}. \quad (1.16)$$

$x$ is called a boundary point of $S$ every ball centered at $x$ intersect both $S$ and $S^c$, i.e.,

$$\forall r > 0, B(r, x) \cap S \neq \emptyset \text{ and } B(r, x) \cap S^c \neq \emptyset. \quad (1.17)$$

We denote by $\partial S$ the set of all boundary points of $S$ called the boundary of $S$:

$$\partial S = \{x \in \mathbb{R}^n : \forall r > 0, B(r, x) \cap S \neq \emptyset \text{ and } B(r, x) \cap S^c \neq \emptyset\}. \quad (1.18)$$

The closure of $S$ is $\bar{S} = S \cup \partial S$.

$S$ is a neighborhood of $x$ if $x$ is an interior point of $S$. 

Definition 1.7. Let $S$ be a subset of $\mathbb{R}^n$.

$S$ is called open if it contains none of its boundary points: $S \cap \partial S = \emptyset$.

$S$ is called closed if it contains all of its boundary points: $\partial S \subset S$.

Note: $\mathbb{R}^n$ and the empty set $\emptyset$ are both open and closed.

Two sets $A$ and $B$ are said to be disjoint if $A \cap B = \emptyset$.

Proposition 1.8. Let $S$ be a subset of $\mathbb{R}^n$. Then

a. $S$ and its complement $S^c$ have the same boundary: $\partial S = \partial (S^c)$.

b. $S^{\text{int}}, \partial S, (S^c)^{\text{int}}$ are mutually disjoint, i.e., $S^{\text{int}} \cap \partial S, (S^c)^{\text{int}} \cap S^{\text{int}}, \partial S \cap (S^c)^{\text{int}}$ are empty sets.

c. $\mathbb{R}^n = S^{\text{int}} \cup \partial S \cup (S^c)^{\text{int}}$.

Consequently, every point $x \in \mathbb{R}^n$ belongs to exactly one of the following sets $S^{\text{int}}, \partial S, (S^c)^{\text{int}}$.

We also have $S \subset S^{\text{int}} \cup \partial S$, hence $\bar{S} = S^{\text{int}} \cup \partial S$, therefore

Proposition 1.9. $(\bar{S})^c = (S^c)^{\text{int}}$.

Proposition 1.10. Suppose $S \subset \mathbb{R}^n$.

a. $S$ is open $\iff$ every point of $S$ is an interior point of $S$ $\iff$ $S = S^{\text{int}}$.

b. $S$ is closed $\iff$ $S^c$ is open.

Proposition 1.11. (i) If $S_1$ and $S_2$ are both open (or closed), so are $S_1 \cup S_2$ and $S_1 \cap S_2$.

(ii) If $\{S_\alpha\}_{\alpha \in I}$ is a family of open sets, then $\bigcup_{\alpha \in I} S_\alpha$ is open.

(iii) If $\{S_\alpha\}_{\alpha \in I}$ is a family of closed sets, then $\bigcap_{\alpha \in I} S_\alpha$ is closed.
1.3 Limits and continuity

Let $n$ and $k$ be two natural numbers. Let $f$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^k$, $a \in \mathbb{R}^n$ and $L \in \mathbb{R}^k$. We say the limit of $f(x)$ as $x$ approaches $a$ is $L$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}^n : 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$  \hspace{1cm} (1.19)

Notation:

$$\lim_{x \to a} f(x) = L.$$ \hspace{1cm} (1.20)

**Proposition 1.12.** The limit $\lim_{x \to a} f(x)$, if exists, is unique.

Some equivalent statements of (1.19):

- If $a = (a_1, a_2, \ldots, a_n)$, then we have $\lim_{x \to a} f(x) = L$ if and only if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : 0 < |x - a| < \max\{|x_1 - a_1|, |x_2 - a_2|, \ldots, |x_n - a_n|\} < \delta \implies |f(x) - L| < \varepsilon.$$ \hspace{1cm} (1.21)

- If $f = (f_1, f_2, \ldots, f_k)$ and $L = (L_1, L_2, \ldots, L_k)$, where each $f_j$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$ then

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a} f_j(x) = L_j \text{ for all } j = 1, 2, \ldots, k.$$ \hspace{1cm} (1.22)

**Example 1.13.** See text, p. 14, 15.

**Proposition 1.14.** Let $f, g: \mathbb{R}^n \to \mathbb{R}^m$, $a \in \mathbb{R}^n$ and

$$\lim_{x \to a} f(x) = L, \quad \lim_{x \to a} g(x) = K.$$ \hspace{1cm} (1.23)

Then

(i) $\lim_{x \to a} (f + g)(x) = L + K$.

In the case $m = 1$, we have

(ii) $\lim_{x \to a} (fg)(x) = LK$.

(iii) If $L \neq 0$, then

$$\lim_{x \to a} \frac{g(x)}{f(x)} = \frac{K}{L}.$$
Remark 1.15. We have
\[
\lim_{x \to a} = L \text{ if and only if } \lim_{x \to a} |f(x) - L| = 0. \tag{1.24}
\]
When \(L = 0\), it becomes
\[
\lim_{x \to a} f(x) = 0 \text{ if and only if } \lim_{x \to a} |f(x)| = 0. \tag{1.25}
\]

Proposition 1.16 ("squeezing property"). Let \(f, g, h : \mathbb{R}^n \to \mathbb{R}\) satisfying \(g(x) \leq f(x) \leq h(x)\) for all \(x \in \mathbb{R}^n\). Suppose \(a \in \mathbb{R}^n\) and
\[
\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L \in \mathbb{R}^m.
\]
Then \(\lim_{x \to a} f(x) = L\).

Proposition 1.17. Let \(f : \mathbb{R}^n \to \mathbb{R}, a \in \mathbb{R}^n\) and \(\lim_{x \to a} f(x) = L\).

(i) If \(f(x) \leq M\) for all \(x \in B(r, a)\) for some \(r > 0\) then \(L \leq M\).

(ii) If \(f(x) \geq m\) for all \(x \in B(r, a)\) for some \(r > 0\) then \(L \geq m\).

Definition 1.18. Let \(a \in \mathbb{R}^n\), we say \(f\) is continuous at \(a\) if
\[
\lim_{x \to a} f(x) = f(a), \tag{1.26}
\]
equivalently,
\[
\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}^n : |x - a| < \delta \implies |f(x) - L| < \varepsilon. \tag{1.27}
\]

Let \(U\) be a subset of \(\mathbb{R}^n\). We say \(f\) is continuous on \(U\) if \(f\) is continuous at every point \(a\) of \(U\).

Proposition 1.19. Let \(U \subset \mathbb{R}^n\) and \(f, g : \mathbb{R}^n \to \mathbb{R}^m\) be continuous on \(U\). Then \((f + g)\) and \((f \cdot g)\) are continuous on \(U\).

In the case \(m = 1\), we have \((fg)\) is continuous on \(U\) and \((f/g)\) is continuous on \(V = U \setminus g^{-1}(\{0\}) = \{x \in U : g(x) \neq 0\}\).

Theorem 1.20. Let \(f : \mathbb{R}^n \to \mathbb{R}^k, g : \mathbb{R}^k \to \mathbb{R}^m,\) and \(U \subset \mathbb{R}^n\). If \(f\) is continuous on \(U\) and \(g\) is continuous on \(f(U)\) then \(g \circ f\) is continuous on \(U\).

Theorem 1.21. Let \(f : \mathbb{R}^n \to \mathbb{R}^m\) be continuous and \(U\) be a subset of \(\mathbb{R}^m\). If \(U\) is open (resp. closed), then \(f^{-1}(U)\) is open (resp. closed).
1.4 Sequences

Let $A$ be a non-empty set. A sequence in $A$ is a function $f : \mathbb{N} \to A$, that is, for all $k \in \mathbb{N}$, $x_k = f(k) \in A$. Notation $\{x_k\}$, $\{x_k\}_1^\infty$, $\{x_k\}_{k=1}^\infty$, . . .

**Definition 1.22.** Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$ and $L \in \mathbb{R}^n$. We say $\{x_k\}$ converges to the limit $L$ if

$$\forall \varepsilon > 0, \exists K \in \mathbb{N}, \forall k \in \mathbb{N} : k > K \implies |x_k - L| < \varepsilon.$$  \hspace{1cm} (1.28)

Notation:

$$\lim_{k \to \infty} x_k = L.$$

In this case, we say the sequence is *convergent*, otherwise the sequence is *divergent*.

In the case $m = 1$ we have the following two definitions

$$\lim_{k \to \infty} x_k = \infty \iff \forall M > 0, \exists K \in \mathbb{N}, \forall k \in \mathbb{N} : k > K \implies x_k > M,$$ \hspace{1cm} (1.29)

$$\lim_{k \to \infty} x_k = -\infty \iff \forall M > 0, \exists K \in \mathbb{N}, \forall k \in \mathbb{N} : k > K \implies x_k < -M.$$ \hspace{1cm} (1.30)

If $\lim_{k \to \infty} x_k = \infty$ or $-\infty$ then $\{x_k\}$ is divergent.

Limits of sequences have similar properties to those of limits of functions.

**Theorem 1.23.** Suppose $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then $x$ belongs to the closure of $S$ if and only if there is a sequence in $S$ converging to $x$.

**Corollary 1.24.** Let $S$ be a subset of $\mathbb{R}^n$. Then $S$ is closed if and only if for every sequence $\{x_k\}$ in $S$ which converges to $a \in \mathbb{R}^n$, we have $a \in S$.

**Theorem 1.25.** Let $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}^m$ and $a \in S$. Then the following are equivalent

a. $f$ is continuous at $a$.

b. For any sequence $\{x_k\}$ in $S$ that converges to $a$, the sequence $\{f(x_k)\}$ converges to $f(a)$. 
Let \( \{x_k\}_{k=1}^\infty \) be a sequence. Let \( k_j \) be a strictly increasing function from \( \mathbb{N} \) to \( \mathbb{N} \), that is, \( k_j \in \mathbb{N} \) for all \( j \in \mathbb{N} \) and \( k_j > k_l \) whenever \( j > l \). Note that the latter property is equivalent to \( k_{j+1} > k_j \) for all \( j \in \mathbb{N} \). Then the sequence \( \{x_{k_j}\}_{j=1}^\infty \) is called a subsequence of \( \{x_k\} \).

**Lemma 1.26.** Let \( k_j \) be a strictly increasing function from \( \mathbb{N} \) to \( \mathbb{N} \). Then \( k_j \geq j \) for all \( j \in \mathbb{N} \).

**Proposition 1.27.** Let \( \{x_k\}_{k=1}^\infty \) be a convergent sequence in \( \mathbb{R}^n \). Then any subsequence \( \{x_{k_j}\}_{j=1}^\infty \) of \( \{x_k\} \) is convergent and

\[
\lim_{j \to \infty} x_{k_j} = \lim_{k \to \infty} x_k.
\]
1.5 Complete\textit{s}e\textit{ness}

Let $S \subset \mathbb{R}$ and $c \in \mathbb{R}$.

- $c$ is an \textit{upper bound} of $S$ if $\forall x \in S, x \leq c$.

- $S$ is said to be \textit{bounded (from) above} if it has an upper bound.

- $c$ is a \textit{lower bound} of $S$ if $\forall x \in S, x \geq c$.

- $S$ is said to be \textit{bounded (from) below} if it has a lower bound.

- We say $S$ is \textit{bounded} if it is bounded above and below, equivalently there are $m, M \in \mathbb{R}$ such that $m \leq x \leq M$ for all $x \in S$, or equivalently, there is $C > 0$ such that $|x| \leq C$ for all $x \in S$.

- A \textit{least upper bound} of $S$, called $\sup S$, is an upper bound of $S$ and is smallest among the all upper bounds of $S$.

- A \textit{greatest lower bound} of $S$, called $\inf S$, is a lower bound of $S$ and is largest among the all lower bounds of $S$.

Note that if $\sup S$ (or $\inf S$) exists then it is unique.

Let $A \subset B \subset \mathbb{R}$. Then

$$\sup A \leq \sup B, \quad \inf B \leq \inf A.$$ \hfill (1.31)

Let $A \subset \mathbb{R}$. Let $B = \{-x : x \in A\}$. If $\sup A$ (resp. $\inf A$) exists then

$$\inf B = -\sup A \quad (\text{resp. } \sup B = -\inf A).$$ \hfill (1.32)

\textbf{Proposition 1.28.} Let $S \subset \mathbb{R}$. Then

$$a = \sup S \iff \begin{cases} (i) \ \forall x \in S, x \leq a, \\ (ii) \ \forall \varepsilon > 0, \exists x_0 \in S : a - \varepsilon < x_0. \end{cases}$$

$$a = \inf S \iff \begin{cases} (i) \ \forall x \in S, x \geq a, \\ (ii) \ \forall \varepsilon > 0, \exists x_0 \in S : x_0 < a + \varepsilon. \end{cases}$$
Remark 1.29. From Proposition 1.28 we see that if \( a = \sup S \) or \( a = \inf S \) then there is a sequence in \( S \) converging to \( a \).

The Completeness Axiom. Let \( S \) be a non-empty subset of \( \mathbb{R} \) which is bounded above, then \( \sup S \) exists.

Corollary 1.30. Let \( S \) be a non-empty subset of \( \mathbb{R} \) which is bounded below, then \( \inf S \) exists.

Definition 1.31. Let \( \{ x_k \} \) be a sequence in \( \mathbb{R} \).

- \( \{ x_k \} \) is increasing if \( x_k \geq x_j \) whenever \( k > j \), or equivalently, \( x_{k+1} \geq x_k \) for all \( k \).

- \( \{ x_k \} \) is decreasing if \( x_k \leq x_j \) whenever \( k > j \), or equivalently, \( x_{k+1} \leq x_k \) for all \( k \).

- \( \{ x_k \} \) is monotone if it is increasing or decreasing.

- \( \{ x_k \} \) is bounded above if the set \( \{ x_k : k \in \mathbb{N} \} \) is bounded above, that is, there is \( M \in \mathbb{R} \) such that \( x_k \leq M \) for all \( k \).

- \( \{ x_k \} \) is bounded below if the set \( \{ x_k : k \in \mathbb{N} \} \) is bounded below, that is, there is \( m \in \mathbb{R} \) such that \( x_k \geq m \) for all \( k \).

- \( \{ x_k \} \) if bounded if it is bounded above and below, equivalently, there is \( C > 0 \) such that \( |x_k| < C \) for all \( k \).

Theorem 1.32. Every bounded monotone sequence in \( \mathbb{R} \) is convergent. More precisely,

(i) If \( \{ x_k \} \) is increasing and bounded above then

\[
\lim_{k \to \infty} x_k = \sup \{ x_k : k \in \mathbb{N} \}. \tag{1.33}
\]

(ii) If \( \{ x_k \} \) is decreasing and bounded below then

\[
\lim_{k \to \infty} x_k = \inf \{ x_k : k \in \mathbb{N} \}. \tag{1.34}
\]
1.5. COMPLETENESS

Theorem 1.33 (The nested interval theorem). Let \( I_k = [a_k, b_k] \) for \( k \in \mathbb{N} \), \( a_k, b_k \in \mathbb{R}, a_k \leq b_k \), be a sequence of intervals that satisfy

(a) \( I_1 \supset I_2 \supset I_3 \supset \ldots \), that is, \( I_k \supset I_{k+1} \) for all \( k \).

(b) \( \lim_{k \to \infty} (b_k - a_k) = 0 \).

Then \( \bigcap_{k=1}^{\infty} I_k = \{c\} \) for some \( c \in \mathbb{R} \).

Using the nested interval theorem, we can prove

Theorem 1.34. Every bounded sequence in \( \mathbb{R} \) has a convergent subsequence.

As a consequence, we have

Theorem 1.35. Every bounded sequence in \( \mathbb{R}^n \) has a convergent subsequence.

Proposition 1.36. Let \( \{x_k\} \) be a convergent sequence in \( \mathbb{R}^n \). Then

(a) \( \{x_k\} \) is bounded.

(b) roughly speaking, \( (x_k - x_j) \to 0 \) as \( k, j \to \infty \); more precisely,

\[
\forall \varepsilon > 0, \exists K \in \mathbb{N}, \forall k \in \mathbb{N}, \forall j \in \mathbb{N} : [(k > K) \land (j > K)] \implies |x_k - x_j| < \varepsilon.
\]

(1.35)

Definition 1.37. A sequence in \( \mathbb{R}^n \) is called a Cauchy sequence if it satisfies (1.35).

Proposition 1.38. Let \( \{x_k\} \) be a Cauchy sequence in \( \mathbb{R}^n \). Then it is bounded.

If, in addition, it has a convergent subsequence \( \{x_{k_j}\}_{j=1}^{\infty} \) then \( \{x_k\} \) itself is convergent and \( \lim_{k \to \infty} x_k = \lim_{j \to \infty} x_{k_j} \).

Combining Theorem 1.35, Propositions 1.36 and 1.38, we obtain

Theorem 1.39. A sequence in \( \mathbb{R}^n \) is convergent if and only if it is Cauchy.
1.6 Compactness

**Definition 1.40.** A subset in $\mathbb{R}^n$ is called *compact* if it is closed and bounded.

**Theorem 1.41** (The Bozano-Weierstrass Theorem). Let $S$ be a subset of $\mathbb{R}^n$. Then the following are equivalent

(a) $S$ is compact

(b) Every sequence in $S$ has a subsequence converging to a point which belongs to $S$.

The relation between compact sets and continuous functions:

**Theorem 1.42.** Let $S \subset \mathbb{R}^n$ be compact and $f : S \to \mathbb{R}^m$ be continuous. Then $f(S)$ is compact (as a subset of $\mathbb{R}^m$).

**Corollary 1.43.** Let $S \subset \mathbb{R}^n$ be compact and $f : S \to \mathbb{R}^m$ be continuous.

**Definition 1.44.** Let $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}$, and $a \in S$.

- $f(a)$ is the *maximum* (largest value) of $f$ on $S$ if $f(a) \geq f(x)$ for all $x \in S$.
- $f(a)$ is the *minimum* (smallest value) of $f$ on $S$ if $f(a) \leq f(x)$ for all $x \in S$.

**Theorem 1.45** (The Extreme Value Theorem). Let $S \subset \mathbb{R}^n$ be compact and $f : S \to \mathbb{R}^m$ be continuous. Then there are $a, b \in S$ such that $f(a)$ is the maximum value of $f$ on $S$ and $f(b)$ is the minimum value of $f$ on $S$. 
1.7. CONNECTEDNESS

1.7 Connectedness

Let $S$ be a subset of $\mathbb{R}^n$.

- $S$ is disconnected if there are non-empty sets $S_1$ and $S_2$ such that

$$S = S_1 \cup S_2, \quad S_1 \cap \bar{S}_2 = \emptyset, \quad S_2 \cap \bar{S}_1 = \emptyset. \quad (1.36)$$

We call the above pair $(S_1, S_2)$ a disconnection of $S$. (Note: they are not unique.)

- $S$ is connected if it is NOT disconnected.

**Theorem 1.46.** The connected subsets of $\mathbb{R}$ are the intervals, i.e.,

$$[a, b], [a, b], (a, b], (a, b), [c, \infty), (c, \infty), (\infty, c), (\infty, c].$$

**Proof.** Skipped (see text). \qed

**Notes:** $S$ is an interval in $\mathbb{R}$ if and only if

$$\forall x, y \in S, \forall z \in \mathbb{R} : x < z < y \implies z \in S. \quad (1.37)$$

**Theorem 1.47.** If $S \subset \mathbb{R}^n$ is connected and $f : S \to \mathbb{R}^m$ is continuous, then $f(S)$ is connected.

**Proof.** Proof by Contraposition: $f(S)$ being disconnected implies $S$ being disconnected.

Suppose $f(S)$ is disconnected then it has a disconnection $(U_1, U_2)$. Let $S_1 = f^{-1}(U_1) = \{x \in S : f(x) \in U_1\}$ and $S_2 = f^{-1}(U_2) = \{x \in S : f(x) \in U_1\}$. Then $S_1, S_2$ are not empty and $S_1 \cup S_2 = S$. Suppose $S_1 \cap \bar{S}_2 \neq \emptyset$, then there is $x_0 \in S_1 \cap \bar{S}_2$. There is a sequence $\{x_k\}$ in $S_2$ such that $x_k \in S_2$, $x_k \to x_0$ as $k \to \infty$. Since $f$ is continuous at $x_0 \in S$: $\lim_{k \to \infty} f(x_k) = f(x_0)$. Note that $f(x_k) \in U_2$, then $f(x_0) \in \bar{U}_2$. But we also have $x_0 \in S_1$ which implies $f(x_0) \in U_1$, therefore $f(x_0) \in U_1 \cap \bar{U}_2$. This contradicts the fact that $U_1 \cap \bar{U}_2 = \emptyset$. Thus $S_1 \cap \bar{S}_2 = \emptyset$. Similarly, $S_2 \cap \bar{S}_1 = \emptyset$. Hence $S$ is disconnected. \qed
Corollary 1.48 (The intermediate value theorem). Suppose $S$ is connected and $f : S \to \mathbb{R}$ is continuous. If $a, b \in S$, $t \in \mathbb{R}$ and $f(a) < t < f(b)$, then there is $c \in S$ such that $f(c) = t$.

Proof. We have $f(S)$ is a connected subset of $\mathbb{R}$, hence it is an interval. Since $f(a), f(b) \in f(S)$, then we have the whole interval $[f(a), f(b)]$ is contained in $f(S)$. Therefore $t \in f(S)$, which means that there is $c \in S$ such that $t = f(c)$. $\square$

Definition 1.49. A set $S \subset \mathbb{R}^n$ is said to be arcwise connected (or pathwise connected) if any two points in $S$ can be joined by a continuous curve in $S$, that is for any $a, y \in S$, there is a continuous function $g : [0, 1] \to S$ such that $g(0) = a$ and $g(1) = b$.

Theorem 1.50. If $S$ is arcwise connected, then $S$ is connected.

Proof. Let $S$ be arcwise connected. Suppose $S$ is disconnected. Let $(S_1, S_2)$ be a disconnection of $S$. There are $a \in S_1$ and $b \in S_2$. Since $S$ is arcwise connected there is a continuous function $f : [0, 1] \to S$ such that $f(0) = a$ and $f(1) = b$. Note that $T = f([0, 1])$ is connected. Let $T_1 = S_1 \cap T$ and $T_2 = S_2 \cap T$. Then $T_1, T_2$ are non-empty sets (containing $a, b$ respectively.). We have $T_1 \cap \bar{T}_2 \subset S_1 \cap \bar{S}_2 = \emptyset$, hence $T_1 \cap \bar{T}_2 = \emptyset$. Similarly, $T_2 \cap \bar{T}_1 = \emptyset$. Therefore, $T$ is disconnected, contradiction. Conclusion: $S$ is connected. $\square$

Let $a, b, c \in S$. If there is a continuous curve in $S$ connecting $a$ and $b$, and one connecting $b$ and $c$, then there is one connecting $a$ and $c$ (transitive relation). Indeed, let $f, g : [0, 1] \to S$ such that $f(0) = a, f(1) = b$ and $g(0) = b, g(1) = c$. Then let $h : [0, 1] \to S$,

$$h(t) = \begin{cases} f(2t) & \text{if } 0 \leq t < 1/2, \\ g(2(1-t)) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

(Verify the continuity of $h$ at $1/2$ using left and right limits.)
Example 1.51. Balls, spheres in \( \mathbb{R}^3 \) and disks, circles in \( \mathbb{R}^2 \) are arcwise-connected, hence connected.

Example 1 p.34 in the text. In \( \mathbb{R}^2 \), let \( a = (-1,0), b = (1,0) \) and \( S_1 = B(1,a), S_2 = B(1,b) \). Let \( S = S_1 \cup S_2 \) and \( T = S_1 \cap \bar{S}_2 \). Then \( S \) is disconnected. Since every point in \( T \) can be connected to the origin \((0,0) \in T\), we have \( T \) is arcwise connected, hence connected.

Note: A connected set is not necessarily arcwise connected. See text p.37 for an example of a set in \( \mathbb{R}^2 \) which is connected but NOT arcwise-connected.

Theorem 1.52. If \( S \) is connected and open, then \( S \) is arcwise connected.

Proof. Let \( S \) be open and connected. Let \( a \) be a fixed point in \( S \). We will prove that we can connect \( a \) to any other points of \( S \), hence showing that \( S \) is arcwise connected.

Set \( S_1 = \{ x \in S : x \text{ is joined by a continuous curve in } S \} \).

Claim: \( S_1 = S \). Then \( S \) is arcwise connected.

Proof of the claim: Suppose \( S_1 \neq S \). Then \( S_2 = S \setminus S_1 \) is not empty and \( S = S_1 \cup S_2 \). Note: \( S_1 \neq \emptyset \) and \( S_1 \cap S_2 = \emptyset \). We now show that \( S_1 \cap \bar{S}_2 \) and \( S_2 \cap \bar{S}_1 \) are empty.

Let \( x \in S_1 \), \( S \) being open implies there is a ball \( B(r,x) \subset S \), \( r > 0 \). For every \( y \in B \), there is a curve from \( a \) to \( x \) then \( x \) to \( y \), hence \( y \in S_1 \). Therefore \( B(1,x) \) is a subset of \( S_1 \). Thus \( x \notin \bar{S}_2 \). We then have \( S_1 \cap \bar{S}_2 = \emptyset \).

Let \( x \in S_2 \), there is a ball \( B = B(r,x) \subset S \). Suppose \( x \in \bar{S}_1 \) then there is \( y \in B \cap S_1 \), hence we can find a continuous curve in \( S \) from \( a \) to \( y \) then \( y \) to \( x \). Thus \( x \in S_1 \), which is absurd since \( x \notin S_1 \) \( (S_1 \cap S_2 = \emptyset) \). Hence \( x \notin \bar{S}_1 \), therefore \( S_2 \cap \bar{S}_1 = \emptyset \).

We have proved \((S_1,S_2)\) is a disconnection of \( S \), which is impossible since \( S \) is connected. Therefore the claim is true and the proof of the theorem is complete. \qed
1.8 Uniform continuity

Let $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$ be continuous. We have

$$\forall x \in S, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in S : |y - x| < \delta \implies |f(x) - f(y)| < \varepsilon. \quad (1.38)$$

The above $\delta$ in general depends on $x, \varepsilon$. In some cases, $\delta$ is independent of $x$, then roughly speaking, the rate $f(y)$ approaches $f(x)$ as $y$ approaches $x$ is controlled uniformly on the whole domain $S$.

**Definition 1.53.** A function $f : S \to \mathbb{R}^m$ is uniformly continuous on $S$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in S, \forall y \in S : |y - x| < \delta \implies |f(x) - f(y)| < \varepsilon. \quad (1.39)$$

**Example 1.54.** The function $f(x) = x^2$ is not uniformly continuous on $(0, \infty)$. Suppose it is, let $\varepsilon > 0$, then there is $\delta > 0$ such that for any $x, y \in (0, \infty)$ and $\delta > 0$, we have

$$|y^2 - x^2| = |y - x||y + x| < \varepsilon.$$

Take $y = x + \delta$ then $2\delta x < \varepsilon$. So $\delta < \varepsilon/(2x)$ which goes to zero as $x$ goes to infinity which is a contradiction since $\delta$ is a fixed positive number.

**Example 1.55.** The function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$. Indeed, by the Mean Value Theorem (next chapter), $|f(x) - f(y)| = |x - y||\cos z| \leq |x - y|$, where $z \in [x, y]$ or $[y, x]$. We can take $\delta = \varepsilon$ in (1.39).

**Example 1.56.** The function $f(x) = x^2$ is uniformly continuous on every bounded subsets of $\mathbb{R}$. Suppose there is $M > 0$ such that $|x| \leq M$ for all $x \in S$. Then for any $x, y \in S$.

$$|f(x) - f(y)| = |x - y||x + y| \leq 2M|x - y|.$$  

We can take $\delta = \varepsilon/(2M)$ in (1.39). *Note:* We can use the Mean Value Theorem as well.

**Theorem 1.57.** Suppose $S$ is compact and $f : S \to \mathbb{R}^m$ is continuous. Then $f$ is uniformly continuous.
1.8. **UNIFORM CONTINUITY**

**Proof.** By contradiction. Suppose $f$ is not uniformly continuous, then

$$\exists \varepsilon_0 > 0, \forall \delta > 0, \exists x, y \in S : |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon_0. \quad (1.40)$$

Take $\delta = 1/k \to 0$. There are sequences $\{x_k\}, \{y_k\}$ in $S$ such that

$$|x_k - y_k| < \frac{1}{k}, \quad |f(x_k) - f(y_k)| \geq \varepsilon_0. \quad (1.41)$$

Since $S$ is compact, there exist convergent subsequences $\{x_{k_j}\}, \{y_{k_j}\}$ whose limits belong to $S$. By the first property of (1.41), we have

$$\lim_{j \to \infty} x_{k_j} = \lim_{j \to \infty} y_{k_j} = x_0 \in S.$$ 

Since $f$ is continuous at $x_0$, $\lim_{j \to \infty} |f(x_{k_j}) - f(y_{k_j})| = |f(x_0) - f(x_0)| = 0$ which contradicts the second property in (1.41). We conclude that $f$ must be uniformly continuous.