Confidence Intervals for the Variance of a Normal Population

MATH 3342
Section 7.4

The Distribution of $S^2$

- Let $X_1, X_2, ..., X_n$ be a random sample from a Normal population with parameters $\mu$ and $\sigma^2$.
- Then the RV
  \[
  \frac{(n-1)S^2}{\sigma^2} = \frac{\sum (X_i - \bar{X})^2}{\sigma^2}
  \]
- Has a chi-squared ($\chi^2$) distribution with $n - 1$ df
Critical Values

- Let $\chi_{\alpha,\nu}^2$ be the chi-squared critical value
- Denotes the number on the horizontal axis such that $\alpha$ of the area under the chi-squared curve with df $\nu$ lies to the right of $\chi_{\alpha,\nu}^2$
- Values are shown in Table A.7 in the Appendix

Derivation of a CI

$$P\left(\frac{(n-1)S^2}{\sigma^2} < \chi_{(\frac{\alpha}{2}),n-1}^2 < \chi_{(1-\frac{\alpha}{2}),n-1}^2\right) = 1-\alpha$$

This is equivalent to:

$$P\left(\frac{(n-1)S^2}{\chi_{(1-\frac{\alpha}{2}),n-1}^2} < \sigma^2 < \frac{(n-1)S^2}{\chi_{(\frac{\alpha}{2}),n-1}^2}\right) = 1-\alpha$$
A 100(1 - \alpha \%)% CI for \sigma^2

- A 100(1 - \alpha \%)% CI for \sigma^2 \text{ is as follows:}

\[
\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-(\alpha/2),n-1}}
\]

- A 100(1 - \alpha \%)% CI for \sigma \text{ is as follows:}

\[
\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}} < \sigma < \sqrt{\frac{(n-1)s^2}{\chi^2_{1-(\alpha/2),n-1}}}
\]

Example

- The amount of lateral expansion was determined for \( n = 12 \) arc welds used in ship containment tanks.
- The resulting sample standard deviation was \( s = 3.56 \) mils.
- Assuming normality, derive a 95% CI for \sigma^2.