Computing Fourier series with *Mathematica*

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This notebook shows how to use *Mathematica* to automate the computation of partial sums of a Fourier series. We restrict ourselves to the case where the function has even symmetry about zero, so that the Fourier series has only cosine terms (you will have a homework problem in which you generalize this calculation to use both sines and cosines). A Fourier series can be defined on different intervals such as [0, 2π] or [−π, π]. Here we use [−π, π].

The Fourier series for an even function will be

\[ f(x) = \sum_{m=0}^{\infty} f_m \cos m x \]

where the coefficients are given by the ratio of inner products

\[ f_m = \frac{\langle \cos m x, f(x) \rangle}{\langle \cos m x, \cos m x \rangle}. \]

The inner product is \( \langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx \).

In approximate calculations we do not take the sum to \( \infty \), but truncate at some finite \( M \):

\[ f(x) \approx \sum_{m=0}^{M} f_m \cos m x. \]

Preliminary setup

Here we define some *Mathematica* functions we’ll use to compute Fourier series: the basis functions, the inner product, the coefficients, and the \( M \)-th partial sum of the series.

- **Define the basis functions**

  The even Fourier basis consists of the cosine functions

  \[
  \phi_{m, x} = \cos(m x)
  \]

- **Define an inner product**

  The inner product appropriate to Fourier analysis on periodic extensions of \([−\pi, \pi]\) is

  \[ \langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx. \]

  (Technical *Mathematica* programming note: the inner product can’t be evaluated yet, because the arguments \( f \) and \( g \) are dummy arguments, not yet defined. The ":=" used in the definition of fourierIP tells *Mathematica* to define fourierIP but to defer evaluation until the function fourierIP is actually used.)

  \[
  \text{fourierIP}[f, g] := \text{Integrate}[f g, \{x, -\pi, \pi\}]
  \]

  Here we show the form of the inner product:
Here's wave. We'll define the Fourier coefficients for a function "func". Note the use of deferred evaluation.

\[
\text{fourierCoeff}[\text{func}, m_] := \text{fourierIP}[\text{func}[x], \phi[m, x]] / \text{fourierIP}[\phi[m, x], \phi[m, x]]
\]

Show the general form of the Fourier coefficient:

\[
\text{fourierCoeff}[\psi, m]
\]

\[
\int_0^\pi \cos(mx) \psi(x) \, dx
\]

\[
\frac{\sin(2m\pi)}{2m} + \pi
\]

- **Sum terms through order M**

Here we define a function that computes the \(M\)-th partial sum of the Fourier series.

\[
\text{fourierSeries}[M_, \text{func}_0, x_] := \text{Sum}[\text{fourierCoeff}[\text{func}, m] \cos(mx), \{m, 0, M\}]
\]

\[
\text{fourierSeries}[M, \psi, x]
\]

\[
\sum_{m=0}^{M} \frac{\sin(2m\pi)}{2m} + \pi \cos(mx) \psi(x) \, dx
\]

At this point, we’re ready to go!

**Example: Fourier series for a triangle wave**

- **Define an expression for the function we want to expand in a Fourier series**

We’ll approximate a triangle wave by a Fourier series. First we define an expression for one period of a triangle wave.

\[
\text{triangleWave}[x_] = \text{Piecewise}[\{(1 - \text{Abs}[x/\pi], \text{Abs}[x] \leq \pi)\}]
\]

\[
\left\{
1 - \frac{|x|}{\pi} \quad |x| \leq \pi
\right\}
\]

Here’s what the function looks like.
We’ll also want to see the periodic extension. The following function will replicate the triangle pulse over several periods.

```
periodicExtension[func_, nPeriods_] := Sum[func[x + 2 k Pi], {k, -nPeriods, nPeriods}]
```

Now we can plot a few periods of the triangle wave:

```
Plot[periodicExtension[triangleWave, 4], {x, -4 Pi, 4 Pi}, PlotRange -> All]
```

- **Compute the fourier series for the triangle wave**

Let’s first look at the Fourier coefficients.

```
fourierCoeff[triangleWave, m] = 
2 (cos(m Pi) - 1) / m^2 [\frac{\sin(2m Pi)}{2m} + \pi]
```

Next, compute the 2nd, 4th, and 16th partial sums:

```
fs2[x_] = fourierSeries[2, triangleWave, x]
```

```
\frac{4 \cos(x)}{\pi^2} + \frac{1}{2}
```
\[ fs4[x_] = f\text{ourierSeries}[4,\ triangleWave, x] \]
\[
\frac{4\cos(x)}{\pi^2} + \frac{4\cos(3x)}{9\pi^2} + \frac{1}{2}
\]

\[ fs16[x_] = f\text{ourierSeries}[16,\ triangleWave, x] \]
\[
\frac{4\cos(x)}{\pi^2} + \frac{4\cos(3x)}{9\pi^2} + \frac{4\cos(5x)}{25\pi^2} + \frac{4\cos(7x)}{49\pi^2} + \frac{4\cos(9x)}{81\pi^2} + \frac{4\cos(11x)}{121\pi^2} + \frac{4\cos(13x)}{169\pi^2} + \frac{4\cos(15x)}{225\pi^2} + \frac{1}{2}
\]

We can plot the partial sums against \( x \).

\[
\text{Plot[\{fs2[x], fs4[x], fs16[x]\}, \{x, -\pi, \pi\}]}
\]

To see how the accuracy improves with increasing number of terms, plot the difference between the series and the exact triangle wave:

\[
\text{Plot[\{fs2[x] - triangleWave[x], fs4[x] - triangleWave[x], fs16[x] - triangleWave[x]\}, \{x, -\pi, \pi\}]}
\]

Finally, plot the Fourier series over several periods, to show that it automatically "builds in" the periodic extension of the square wave.
Plot[{fs2[x], fs16[x]}, {x, -6 Pi, 6 Pi}]