Solve $y' + 3y = \sin x$ with $y(0) = 1$ with Mathematica

- **Define $p$ and $q$, and compute the integrating factor**

To put the equation in standard form $y' + p(x)y = q(x)$ we set $p(x) = 3$ and $q(x) = \sin x$.

$$p[x_] = 3$$
$$q[x_] = \text{Sin}[x]$$

The integrating factor is $\mu = e^{\int (dx)}$.

$$\mu[x_] = \text{Exp}[\text{Integrate}[p[x], x]]$$
$$e^{3x}$$

- **Define values for the initial conditions $x_0$ and $y_0$**

$$x0 = 0$$
$$0$$

$$y0 = 1$$
$$1$$

- **Use the standard solution formula to compute $y(x)$**

The solution is

$$y(x) = \frac{1}{\mu(x)} \left[ \mu(x_0) y(x_0) + \int_{x_0}^{x} \mu(x) q(x) \, dx \right]$$

which is entered into Mathematica as

$$y[x_] = 1/\mu[x] \left( \mu[x0] y0 + \text{Integrate}[\mu[x] q[x], \{x, x0, x\}] \right)$$
$$e^{-3x} \left( \frac{1}{10} \left( 1 - e^{3x} (\cos(x) - 3 \sin(x)) \right) + 1 \right)$$

Running `Expand[]` multiplies everything out

$$\text{Expand}[y[x]]$$
$$\frac{\cos(x)}{10} + \frac{11 e^{-3x}}{10} + \frac{3 \sin(x)}{10}$$

This is the same answer obtained by hand previously.
Check the solution

Plug \( y(x) \) into the LHS, and compare to the RHS.

\[
\text{LHS} = \text{FullSimplify}[D[y[x], x] + p[x] y[x]] \\
\sin(x)
\]

\[
\text{RHS} = q[x] \\
\sin(x)
\]

These are the same, so LHS=RHS and the equation checks

\[
\text{LHS} = \text{RHS} \\
\text{True}
\]

Verify the solution satisfies the initial conditions

\[
y[x_0] = y_0 \\
\text{True}
\]

All done!