Math 5311 Spr 2011 - Homework #6

Due Monday 4 April

1. Let \(M\) and \(K\) be SPD matrices of identical size and \(a(t)\) a vector-valued function of time. We use the notation \(\dot{a}\) for \(\frac{da}{dt}\) and \(\ddot{a}\) for \(\frac{d^2a}{dt^2}\).

(a) Show that if \(a(t)\) obeys the differential equation
\[
M \ddot{a} + Ka = 0
\]
then the quantity
\[
E(a) = \frac{1}{2} \dot{a}^T M \dot{a} + \frac{1}{2} a^T Ka
\]
is constant in time, i.e., \(\frac{dE}{dt} = 0\).

(b) Show that if \(a(t)\) obeys the differential equation
\[
M \ddot{a} + \gamma M \dot{a} + Ka = 0
\]
with \(\gamma > 0\) then the function \(E\) defined above will obey the inequality \(\frac{dE}{dt} \leq 0\).

2. Consider the second-order ODE
\[
\ddot{y} + y = 0.
\]

(a) Show that this ODE is equivalent to the first-order system
\[
\frac{du}{dt} = Su
\]
with \(u = \begin{pmatrix} y \\ v \end{pmatrix}\) and \(S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\).

(b) Show that \(E(u) = \frac{1}{2} u^T S u\) is constant in time.

(c) Suppose you try to approximate the solution to the system \(\dot{u} = Su\) using the backward Euler method. Let \(u_n\) be the solution \(u(t_n)\) at the \(n\)-th timestep. Prove that
\[
E(u_{n+1}) < E(u_n).
\]

(d) Program the backward Euler method for this problem. Start with \(u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and follow for 5000 timesteps of length \(\Delta t = 0.01\). Plot the trajectory in the \((y, v)\) plane. Interpret this plot in terms of the results of parts (b) and (c). What would an exact solution look like in the \((y, v)\) plane?

(e) The velocity Verlet method for solving an IVP of the form
\[
\begin{align*}
y' &= v \\
v' &= f(y) \\
y(0) &= y_0 \\
v(0) &= v_0
\end{align*}
\]
is the following sequence of steps:
\[
\begin{align*}
y_{n+1} &= y_n + v_n \Delta t + \frac{1}{2} f(y_n) \Delta t^2 \\
v_{n+1} &= v_n + \frac{\Delta t}{2} [f(y_n) + f(y_{n+1})]
\end{align*}
\]
Program the velocity Verlet method for this problem (in which case \(f(y) = -y\)), and repeat part (d) using the velocity Verlet method in place of the backward Euler method.
3. Suppose that at \( x = 0 \) there is an interface between two materials, 1 and 2. In material 1 in the region \( x < 0 \), some physical property \( u_1(x,t) \) obeys the wave equation

\[
\frac{\partial^2 u_1}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u_1}{\partial t^2} = 0
\]

where \( c \) is a wave speed. In material 2 in the region \( x \geq 0 \), a property \( u_2 \) (perhaps the same physical property, but labeled differently because its functional form may be different) obeys a modified wave equation,

\[
\frac{\partial^2 u_2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u_2}{\partial t^2} - \frac{\omega_p^2}{c^2} u = 0
\]

where \( c \) is the same wave speed and \( \omega_p \) is a real constant. Suppose also that at \( x = 0 \), the interface conditions

\[
\begin{align*}
   u_1(0,t) &= u_2(0,t) \quad \forall t \\
   \frac{\partial u_1}{\partial x}(0,t) &= \frac{\partial u_2}{\partial x}(0,t) \quad \forall t
\end{align*}
\]

hold.

(a) Suppose that \( u_1 \) and \( u_2 \) both have sinusoidal variation in time, which we can write as

\[
\begin{align*}
   u_1(x,t) &= v_1(x) e^{-i\omega t} \\
   u_2(x,t) &= v_2(x) e^{-i\omega t}
\end{align*}
\]

i. Show that the choice

\[
v_1(x) = e^{\pm ik_1x}
\]

lets \( u_1 \) be a solution to the wave equation when \( ck_1 = \omega \).

ii. Show that when \( \omega > \omega_p \) the complex exponential

\[
v_2(x) = e^{\pm ik_2x}
\]

lets \( u_2 \) be a solution to the modified equation when \( ck_2 = \sqrt{\omega^2 - \omega_p^2} \).

iii. Show that when \( \omega < \omega_p \) the real exponential

\[
v_2(x) = e^{\pm k_2x}
\]

lets \( u_2 \) be a solution to the modified equation when \( ck_2 = \sqrt{\omega_p^2 - \omega^2} \).

(b) Consider the case \( \omega > \omega_p \), in which case a solution with no wave incoming from the right is

\[
v(x) = \begin{cases} 
  e^{-ik_1x} + Re^{ik_1x} & x < 0 \\
  Te^{-ik_2x} & x > 0
\end{cases}
\]

Apply the interface conditions to compute \( R \) and \( T \).

(c) Consider the case \( \omega < \omega_p \), in which case a solution that is bounded on the right is

\[
v(x) = \begin{cases} 
  e^{-ik_1x} + Re^{ik_1x} & x < 0 \\
  Te^{-ik_2x} & x > 0
\end{cases}
\]

Apply the interface conditions to compute \( R \) and \( T \). Note that \( R \) and \( T \) may be complex.

(d) Take the values \( c = 1, \omega_p = 1 \). Plot the real part of \( v(x) \) on \( x \in [-16\pi, 16\pi] \) for

\[
\omega \in \begin{cases} 
  \frac{1}{2}, \frac{15}{16}, \frac{127}{128} \end{cases}
\]

and

\[
\omega \in \begin{cases} 
  \frac{2}{15}, \frac{16}{128}, \frac{127}{128} \end{cases}
\]