1. For each of the following functionals compute the Gateaux differential $d_v W(u)$
   
   (a) $W(u) = \int_{-1}^{1} e^{u'} \, dx$
   (b) $W(u) = \int_{-1}^{1} (u')^2 + u^2 \, dx$

2. Throughout this problem $H^1_0$ is the space $\{ v : \int_0^1 (v')^2 + v^2 \, dx < \infty \land v(0) = 0 \land v(1) = 0 \}$. For each of the following weak BVPs, and supposing that you have a current guess $u^{(n)}(x) \in H^1_0$, derive a linearized weak BVP for the Newton step $w(x) = u^{(n+1)}(x) - u^{(n)}(x)$.
   
   (a) $\int_0^1 [v'u' + ve^{-u}] \, dx = 0 \ \forall v \in H^1_0$
   (b) $\int_0^1 [(1 + u)^3 v'u' + v] \, dx = 0 \ \forall v \in H^1_0$

3. Let $A = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
   
   (a) Use Gateaux differentiation to derive a necessary condition for the minimization of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

   $$f(x) = \frac{1}{2} x^T A x - x^T b.$$ 

   (b) Solve the resulting equation to find the minimum.
   (c) Make a contour plot of $f$ and check that your computed minimum is consistent with the plot.

4. Let $A$ be any $N \times N$ real symmetric matrix. Prove that the function $f(x) = \frac{1}{2} x^T A x$ has a unique minimum at $x = 0$ iff $A$ is positive definite.