Vector Formulas

In these notes we use notation like $F$ for vector valued functions and we use either

$$F(t) = (f_1(t), f_2(t), f_3(t)) = f_1(t)i + f_2(t)j + f_3(t)k$$

for vector valued functions in $\mathbb{R}^3$ or $F(t) = (f_1(t), f_2(t)) = f_1(t)i + f_2(t)j$ for vector valued functions in $\mathbb{R}^2$. In what follows we will usually give the formulas for $\mathbb{R}^3$. If a formula is only valid in $\mathbb{R}^3$ (such as the cross product we will note this).

Consider vector valued function $F$ as above and $G = (g_1(t), g_2(t), g_3(t)) = g_1(t)i + g_2(t)j + g_3(t)k$ and scalar function $f(t), g(t)$.

1. Scalar Times Vector: $(fG)(t) = f(t)f_1(t)i + f(t)f_2(t)j + f(t)f_3(t)k$
2. Dot Product: $(F \cdot G)(t) = f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t)$
3. Cross Product (only in $\mathbb{R}^3$):

$$\langle F \times G \rangle(t) = \begin{vmatrix} i & j & k \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix}$$

$$= (f_2g_3 - f_3g_2)(t)i - (f_1g_3 - f_3g_1)(t)j + (f_1g_2 - f_2g_1)(t)k$$

4. \[ \lim_{t \to c} F(t) = \langle \lim_{t \to c} f_1(t), \lim_{t \to c} f_2(t), \lim_{t \to c} f_3(t) \rangle \]

5. \[ \frac{dF}{dt}(t) = \left\langle \frac{df_1}{dt}(t), \frac{df_2}{dt}(t), \frac{df_3}{dt}(t) \right\rangle. \]

6. Given a curve $F(t)$ the derivative $T(t) = F'(t)$ is a tangent vector to the curve at $F(t)$ and $T(t)/\|T(t)\|$ is a unit tangent vector.

7. \[ \int F(t) \, dt = \left\langle \int f_1(t) \, dt, \int f_2(t) \, dt, \int f_3(t) \, dt \right\rangle + C \text{ where } C = \langle C_1, C_2, C_3 \rangle. \]

8. \[ \int_a^b F(t) \, dt = \left\langle \int_a^b f_1(t) \, dt, \int_a^b f_2(t) \, dt, \int_a^b f_3(t) \, dt \right\rangle. \]

9. $(fF)' = f'F + fF'$, \hspace{1cm} $(F \cdot G)' = F' \cdot G + F \cdot G'$,

$$(F \times G)' = F' \times G + F \times G'$, \hspace{1cm} (F(f(t)))' = F'(f(t))f'(t).$$

10. Position: $R(t) = f_1(t)i + f_2(t)j + f_3(t)k$; Velocity: $V(t) = R'(t)$; Speed: $\|V(t)\|$; Direction $V(t)/\|V(t)\|$; Acceleration: $A(t) = V'(t)$.

11. Projectile in $\mathbb{R}^2$: $A = -gj$ (\(g = 32 ft/s^2, g = 9.8 m/s^2\)), $V(0) = v_0 \cos(\alpha)i + v_0 \sin(\alpha)j$, $v_0 = \|V(0)\|$, $R(0) = s_0j$. Then $R(t) = [(v_0 \cos(\alpha))t]i + \left[(v_0 \sin(\alpha))t - \frac{1}{2}gt^2 + s_0\right]j$
In order to write vector $\mathbf{r}$ with respect to orthogonal components in polar coordinates we use the unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ which are defined via

$$
\mathbf{u}_r = (\cos(\theta))\mathbf{i} + (\sin(\theta))\mathbf{j}, \quad \mathbf{u}_\theta = (-\sin(\theta))\mathbf{i} + (\cos(\theta))\mathbf{j}
$$

We can readily compute

$$
\frac{d\mathbf{u}_r}{d\theta} = (-\sin(\theta))\mathbf{i} + (\cos(\theta))\mathbf{j} = \mathbf{u}_\theta
$$

and

$$
\frac{d\mathbf{u}_\theta}{d\theta} = (-\cos(\theta))\mathbf{i} + (-\sin(\theta))\mathbf{j} = -\mathbf{u}_r
$$

If we have a radial vector field, i.e., $\mathbf{R}(t) = r\mathbf{u}_r = (r\cos(\theta))\mathbf{i} + (r\sin(\theta))\mathbf{j}$ where $r = \|\mathbf{R}\|$, we can compute $\mathbf{V}$ and $\mathbf{A}$. It is important to notice that $r$ and $\theta$ depend on $t$.

$$
\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt}
$$

$$
= \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta
$$

Similarly we can compute

$$
\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2}
$$

$$
= \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] \mathbf{u}_r + \left[ r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right] \mathbf{u}_\theta
$$