theorem can be formulated to establish the existence of a unique positive \textit{nth} root of \(a\), denoted by \(\sqrt[n]{a}\) or \(a^{1/n}\), for each \(n \in \mathbb{N}\).

\textbf{Remark} If in the proof of Theorem 2.4.7 we replace the set \(S\) by the set of rational numbers \(T := \{r \in \mathbb{Q} : 0 \leq r, r^2 < 2\}\), the argument then gives the conclusion that \(y := \sup T\) satisfies \(y^2 = 2\). Since we have seen in Theorem 2.1.4 that \(y\) cannot be a rational number, it follows that the set \(T\) that consists of rational numbers does not have a supremum belonging to the set \(\mathbb{Q}\). Thus the ordered field \(\mathbb{Q}\) of rational numbers does \textit{not} possess the Completeness Property.

\textbf{Density of Rational Numbers in \(\mathbb{R}\)}

We now know that there exists at least one irrational real number, namely \(\sqrt{2}\). Actually there are "more" irrational numbers than rational numbers in the sense that the set of rational numbers is countable (as shown in Section 1.3), while the set of irrational numbers is uncountable (see Section 2.5). However, we next show that in spite of this apparent disparity, the set of rational numbers is "dense" in \(\mathbb{R}\) in the sense that given any two real numbers there is a rational number between them (in fact, there are infinitely many such rational numbers).

\textbf{2.4.8 The Density Theorem} If \(x\) and \(y\) are any real numbers with \(x < y\), then there exists a rational number \(r \in \mathbb{Q}\) such that \(x < r < y\).

\textbf{Proof}. It is no loss of generality (why?) to assume that \(x > 0\). Since \(y - x > 0\), it follows from Corollary 2.4.5 that there exists \(n \in \mathbb{N}\) such that \(1/n < y - x\). Therefore, we have \(nx + 1 < ny\). If we apply Corollary 2.4.6 to \(nx > 0\), we obtain \(m \in \mathbb{N}\) with \(m - 1 \leq nx < m\). Therefore, \(m \leq nx + 1 < ny\), whence \(nx < m < ny\). Thus, the rational number \(r := m/n\) satisfies \(x < r < y\).

Q.E.D.

To round out the discussion of the interlacing of rational and irrational numbers, we have the same "betweenness property" for the set of irrational numbers.

\textbf{2.4.9 Corollary} If \(x\) and \(y\) are real numbers with \(x < y\), then there exists an irrational number \(z\) such that \(x < z < y\).

\textbf{Proof}. If we apply the Density Theorem 2.4.8 to the real numbers \(x/\sqrt{2}\) and \(y/\sqrt{2}\), we obtain a rational number \(r \neq 0\) (why?) such that

\[ \frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}. \]

Then \(z := r\sqrt{2}\) is irrational (why?) and satisfies \(x < z < y\).

Q.E.D.

\textbf{Exercises for Section 2.4}

1. Show that \(\sup\{1 - 1/n : n \in \mathbb{N}\} = 1\).
2. If \(S := \{1/n - 1/m : n, m \in \mathbb{N}\}\), find \(\inf S\) and \(\sup S\).
3. Let \(S \subseteq \mathbb{R}\) be nonempty. Prove that if a number \(u\) in \(\mathbb{R}\) has the properties: (i) for every \(n \in \mathbb{N}\) the number \(u - 1/n\) is not an upper bound of \(S\), and (ii) for every number \(n \in \mathbb{N}\) the number \(u + 1/n\) is an upper bound of \(S\), then \(u = \sup S\). (This is the converse of Exercise 2.3.9.)
4. Let $S$ be a nonempty bounded set in $\mathbb{R}$.
   (a) Let $a > 0$, and let $aS := \{as : s \in S\}$. Prove that
   \[
   \inf(aS) = a \inf S, \quad \sup(aS) = a \sup S.
   \]
   (b) Let $b < 0$ and let $bS = \{bs : s \in S\}$. Prove that
   \[
   \inf(bS) = b \sup S, \quad \sup(bS) = b \inf S.
   \]

5. Let $S$ be a set of nonnegative real numbers that is bounded above and let $T := \{x^2 : x \in S\}$. Prove that if $u = \sup S$, then $u^2 = \sup T$. Give an example that shows the conclusion may be false if the restriction against negative numbers is removed.

6. Let $X$ be a nonempty set and let $f : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, show that Example 2.4.1(a) implies that
   \[
   \sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}.
   \]
   Show that we also have
   \[
   \inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}.
   \]

7. Let $A$ and $B$ be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.

8. Let $X$ be a nonempty set, and let $f$ and $g$ be defined on $X$ and have bounded ranges in $\mathbb{R}$. Show that
   \[
   \sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}
   \]
   and that
   \[
   \inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x) + g(x) : x \in X\}.
   \]
   Give examples to show that each of these inequalities can be either equalities or strict inequalities.

9. Let $X = Y := \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) := 2x + y$.
   (a) For each $x \in X$, find $f(x) := \sup\{h(x, y) : y \in Y\}$; then find $\inf\{f(x) : x \in X\}$.
   (b) For each $y \in Y$, find $g(y) := \inf\{h(x, y) : x \in X\}$; then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (a).

10. Perform the computations in (a) and (b) of the preceding exercise for the function $h : X \times Y \to \mathbb{R}$ defined by
    \[
    h(x, y) := \begin{cases}
    0 & \text{if } x < y, \\
    1 & \text{if } x \geq y.
    \end{cases}
    \]

11. Let $X$ and $Y$ be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be defined by
    \[
    f(x) := \sup\{h(x, y) : y \in Y\}, \quad g(y) := \inf\{h(x, y) : x \in X\}.
    \]
    Prove that
    \[
    \sup\{g(y) : y \in Y\} \leq \inf\{f(x) : x \in X\}.
    \]
    We sometimes express this by writing
    \[
    \sup_x \inf_y h(x, y) \leq \inf_x \sup_y h(x, y).
    \]
    Note that Exercises 9 and 10 show that the inequality may be either an equality or a strict inequality.
12. Let $X$ and $Y$ be nonempty sets and let $h : X \times Y \rightarrow \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $F : X \rightarrow \mathbb{R}$ and $G : Y \rightarrow \mathbb{R}$ be defined by

$$F(x) := \sup \{h(x, y) : y \in Y\}, \quad G(y) := \sup \{h(x, y) : x \in X\}.$$ 

Establish the Principle of the Iterated Suprema:

$$\sup \{h(x, y) : x \in X, y \in Y\} = \sup \{F(x) : x \in X\} = \sup \{G(y) : y \in Y\}$$

We sometimes express this in symbols by

$$\sup_{x \in X} \sup_{y \in Y} h(x, y) = \sup_{x \in X} h(x, \cdot) = \sup_{y \in Y} h(\cdot, y).$$

13. Given any $x \in \mathbb{R}$, show that there exists a unique $n \in \mathbb{Z}$ such that $n - 1 \leq x < n$.

14. If $y > 0$, show that there exists $n \in \mathbb{N}$ such that $1/2^n < y$.

15. Modify the argument in Theorem 2.4.7 to show that there exists a positive real number $y$ such that $y^2 = 3$.

16. Modify the argument in Theorem 2.4.7 to show that if $a > 0$, then there exists a positive real number $x$ such that $x^2 = a$.

17. Modify the argument in Theorem 2.4.7 to show that there exists a positive real number $u$ such that $u^2 = 2$.

18. Complete the proof of the Density Theorem 2.4.8 by removing the assumption that $x > 0$.

19. If $u > 0$ is any real number and $x < y$, show that there exists a rational number $r$ such that $x < ru < y$. (Hence the set $\{ru : r \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.)

Section 2.5 Intervals

The Order Relation on $\mathbb{R}$ determines a natural collection of subsets called "intervals." The notations and terminology for these special sets will be familiar from earlier courses. If $a, b \in \mathbb{R}$ satisfy $a < b$, then the open interval determined by $a$ and $b$ is the set

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$ 

The points $a$ and $b$ are called the endpoints of the interval; however, the endpoints are not included in an open interval. If both endpoints are adjoined to this open interval, then we obtain the closed interval determined by $a$ and $b$; namely, the set

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$ 

The two half-open (or half-closed) intervals determined by $a$ and $b$ are $(a, b]$ and $[a, b)$, which includes the endpoint $a$, and $(a, b)$, which includes the endpoint $b$.

Each of these four intervals is bounded and has length defined by $b - a$. If $a = b$, the corresponding open interval is the empty set $(a, a) = \emptyset$, whereas the corresponding closed interval is the singleton set $[a, a] = \{a\}$.

There are five types of unbounded intervals for which the symbols $\infty$ (or $+\infty$) and $-\infty$ are used as notational convenience in place of the endpoints. The infinite open intervals are the sets of the form

$$(a, \infty) := \{x \in \mathbb{R} : x > a\} \quad \text{and} \quad (-\infty, b) := \{x \in \mathbb{R} : x < b\}.$$
The first set has no upper bounds and the second one has no lower bounds. Adjoining endpoints gives us the infinite closed intervals:

\[ [a, \infty) := \{ x \in \mathbb{R} : a \leq x \} \quad \text{and} \quad (-\infty, b] := \{ x \in \mathbb{R} : x \leq b \}. \]

It is often convenient to think of the entire set \( \mathbb{R} \) as an infinite interval; in this case, we write \( (-\infty, \infty) := \mathbb{R} \). No point is an endpoint of \( (-\infty, \infty) \).

**Warning** It must be emphasized that \( \infty \) and \( -\infty \) are not elements of \( \mathbb{R} \), but only convenient symbols.

### Characterization of Intervals

An obvious property of intervals is that if two points \( x, y \) with \( x < y \) belong to an interval \( I \), then any point lying between them also belongs to \( I \). That is, if \( x < t < y \), then the point \( t \) belongs to the same interval as \( x \) and \( y \). In other words, if \( x \) and \( y \) belong to an interval \( I \), then the interval \( [x, y] \) is contained in \( I \). We now show that a subset of \( \mathbb{R} \) possessing this property must be an interval.

#### 2.5.1 Characterization Theorem

If \( S \) is a subset of \( \mathbb{R} \) that contains at least two points and has the property

\[
(1) \quad \text{if } x, y \in S \text{ and } x < y, \text{ then } [x, y] \subseteq S,
\]

then \( S \) is an interval.

**Proof.** There are four cases to consider: (i) \( S \) is bounded, (ii) \( S \) is bounded above but not below, (iii) \( S \) is bounded below but not above, and (iv) \( S \) is neither bounded above nor below.

Case (i): Let \( a := \inf S \) and \( b := \sup S \). Then \( S \subseteq [a, b] \) and we will show that \( (a, b) \subseteq S \).

If \( a < z < b \), then \( z \) is not a lower bound of \( S \), so there exists \( x \in S \) with \( x < z \). Also, \( z \) is not an upper bound of \( S \), so there exists \( y \in S \) with \( z < y \). Therefore \( z \in [x, y] \), so property (1) implies that \( z \in S \). Since \( z \) is an arbitrary element of \( (a, b) \), we conclude that \( (a, b) \subseteq S \).

Now if \( a \notin S \) and \( b \notin S \), then \( S = [a, b] \). (Why?) If \( a \notin S \) and \( b \notin S \), then \( S = (a, b) \).

The other possibilities lead to either \( S = (a, b) \) or \( S = [a, b] \).

Case (ii): Let \( b := \sup S \). Then \( S \subseteq (-\infty, b] \) and we will show that \( (-\infty, b] \subseteq S \). For, if \( z < b \), then there exist \( x, y \in S \) such that \( z \in [x, y] \subseteq S \). (Why?) Therefore \( (-\infty, b] \subseteq S \).

If \( b \in S \), then \( S = (-\infty, b] \), and if \( b \notin S \), then \( S = (-\infty, b) \).

Cases (iii) and (iv) are left as exercises. Q.E.D.

### Nested Intervals

We say that a sequence of intervals \( I_n, n \in \mathbb{N} \), is nested if the following chain of inclusions holds (see Figure 2.5.1):

\[
I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots
\]

For example, if \( I_n := [0, 1/n] \) for \( n \in \mathbb{N} \), then \( I_n \supseteq I_{n-1} \) for each \( n \in \mathbb{N} \) so that this sequence of intervals is nested. In this case, the element 0 belongs to all \( I_n \) and the Archimedean Property 2.4.3 can be used to show that 0 is the only such common point. (Prove this.) We denote this by writing \( \bigcap_{n=1}^\infty I_n = \{0\} \).

It is important to realize that, in general, a nested sequence of intervals need not have a common point. For example, if \( I_n := (0, 1/n] \) for \( n \in \mathbb{N} \), then this sequence of intervals is