PROBLEM SET

Problems on Matrix Exponentials

Math 3351, Spring 2011

March 29, 2011

ANSWERS
These problems assume the use of a TI-89 (or similar) calculator. To find the eigenvalues of a matrix, use the calculator to find the characteristic polynomial and use the `solve` or `csolve` functions to find the roots. Use the calculator to find RREFs and powers of matrices.

The eigenvalue and eigenvectors functions on the calculator probably won’t work on these problems because they only give approximate answers.
Express your answers as fractions, not decimals.
In the answers, \( \exp(t) \) is often used in place of \( e^t \)

**Problem 1.**
In each part, you are given a matrix \( A \) which is diagonalizable. Find the diagonalization and use it to find \( e^{tA} \). If an initial value problem is given, find the solution.

**A.**
\[
A = \begin{bmatrix}
2 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

**Answer:**
Of course, this matrix is already diagonal. We just use the rule for finding the exponential of a diagonal matrix,
\[
e^{tA} = \begin{bmatrix}
e^{2t} & 0 & 0 \\
0 & e^{5t} & 0 \\
0 & 0 & e^t
\end{bmatrix}
\]

**B.**
\[
A = \begin{bmatrix}
6 & 4 \\
-3 & -1
\end{bmatrix}
\]

Solve the initial value problem
\[
x'(t) = Ax(t)
\]
\[
x(0) = c = \begin{bmatrix}
-2 \\
3
\end{bmatrix}
\]

**Answer:**
The eigenvalues are 2 and 3.
We can calculate
\[
A - 2I = \begin{bmatrix}
4 & 4 \\
-3 & -3
\end{bmatrix}
\]
The RREF of \( A - 2I \) is
\[
R = \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}
\]
From this, we calculate that the nullspace of $A - 2I$ is one dimensional and is spanned by

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$ 

Next we calculate that

$$A - 3I = \begin{bmatrix} 3 & 4 \\ -3 & -4 \end{bmatrix}.$$ 

The RREF of $A - 3I$ is

$$R = \begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix}.$$ 

The nullspace of $A - 3I$ is one dimensional and is spanned by

$$v_2 = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}.$$ 

We form the matrix

$$P := \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -1 & -4/3 \\ 1 & 1 \end{bmatrix}.$$ 

We can then calculate that

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$ 

Thus, $A$ is diagonalizable. We also have $A = PDP^{-1}$.

Since $D$ is diagonal, we can easily compute

$$e^{tD} = \begin{bmatrix} \exp (2t) & 0 \\ 0 & \exp (3t) \end{bmatrix}.$$ 

The we can compute

$$e^{tA} = e^{tPDP^{-1}} = Pe^{tD}P^{-1} = \begin{bmatrix} -3 \exp (2t) + 4 \exp (3t) & -4 \exp (2t) + 4 \exp (3t) \\ 3 \exp (2t) - 3 \exp (3t) & 4 \exp (2t) - 3 \exp (3t) \end{bmatrix}.$$ 

To solve the initial value problem, we just compute

$$x(t) = e^{tA}c = \begin{bmatrix} -6 \exp (2t) + 4 \exp (3t) \\ 6 \exp (2t) - 3 \exp (3t) \end{bmatrix}.$$
C.

\[
A = \begin{bmatrix}
122 & 0 & -492 \\
66 & -1 & -264 \\
30 & 0 & -121
\end{bmatrix}.
\]

Answer:
The characteristic polynomial is \( p(\lambda) = \lambda^3 - 3\lambda - 2 \) which factors as \( p(\lambda) = (\lambda - 2)(\lambda + 1)^2 \), so the eigenvalues are 2 and -1.

We calculate

\[
A - 2I = \begin{bmatrix}
120 & 0 & -492 \\
66 & -3 & -264 \\
30 & 0 & -123
\end{bmatrix}.
\]

The RREF of \( A - 2I \) is

\[
R = \begin{bmatrix}
1 & 0 & -41/10 \\
0 & 1 & -11/5 \\
0 & 0 & 0
\end{bmatrix}.
\]

From the RREF we calculate that the nullspace of \( A - 2I \) is one dimensional and is spanned by

\[
v_1 = \begin{bmatrix}
41/10 \\
11/5 \\
1
\end{bmatrix}.
\]

Next, we calculate

\[
A - (-1)I = A + I = \begin{bmatrix}
123 & 0 & -492 \\
66 & 0 & -264 \\
30 & 0 & -120
\end{bmatrix}.
\]

The RREF of \( A + I \) is

\[
R = \begin{bmatrix}
1 & 0 & -4 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

from which we calculate that the nullspace of \( A + I \) is two dimensional with basis

\[
v_2 = \begin{bmatrix}
4 \\
0 \\
1
\end{bmatrix}, \quad v_3 = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}.
\]

The diagonalizing matrix is

\[
P = \begin{bmatrix}
v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix}
41/10 & 4 & 0 \\
11/5 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]
The diagonalization of $A$ is

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$  

To compute the exponentials, we first compute

$$e^{tD} = \begin{bmatrix} \exp(2t) & 0 & 0 \\ 0 & \exp(-t) & 0 \\ 0 & 0 & \exp(-t) \end{bmatrix}.$$  

We then have $e^{tA} = P e^{tD} P^{-1}$, so the answer is

$$e^{tA} = \begin{bmatrix} 41 \exp(2t) - 40 \exp(-t) & 0 & -164 \exp(2t) + 164 \exp(-t) \\ 22 \exp(2t) - 22 \exp(-t) & \exp(-t) & -88 \exp(2t) + 88 \exp(-t) \\ 10 \exp(2t) - 10 \exp(-t) & 0 & -40 \exp(2t) + 41 \exp(-t) \end{bmatrix}.$$  

D.

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}.$$  

Solve the initial value problem

$$x(t) = Ax(t)$$

$$x(0) = c = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$  

**Answer:**

The eigenvalues of $A$ are $2 + 3i$ and $2 - 3i$.

We first find a basis for the eigenspace $E(2 + 3i)$. We calculate

$$A - (2 + 3i)I = \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix}.$$  

The RREF of $A - (2 + 3i)$ is

$$R = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}.$$  

From which we calculate that nullspace of $A - (2 + 3i)I$ is one dimensional with basis vector

$$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$  

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Since $2 - 3i$ is the conjugate of $2 + 3i$, we can find a basis of $E(2 - 3i)$ by just taking the conjugate of our basis for $E(2 + 3i)$. Thus, $E(2 - 3i)$ is one dimensional with basis

$$v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$ 

The diagonalizing matrix is then

$$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}.$$ 

The diagonalization of $A$ is

$$D = \begin{bmatrix} 2 + 3i & 0 \\ 0 & 2 - 3i \end{bmatrix}.$$ 

To compute the exponentials, we compute

$$e^{tD} = \begin{bmatrix} \exp((2 + 3i)t) & 0 \\ 0 & \exp((2 - 3i)t) \end{bmatrix}.$$ 

We compute $e^{tA} = Pe^{tD}P^{-1}$. We get

$$e^{tA} = \begin{bmatrix} \exp(2t) \cos(3t) & -\exp(2t) \sin(3t) \\ \exp(2t) \sin(3t) & \exp(2t) \cos(3t) \end{bmatrix}.$$ 

The solution of the given initial value problem is

$$x(t) = e^{tA}c = \begin{bmatrix} -3\exp(2t) \cos(3t) - 2\exp(2t) \sin(3t) \\ -3\exp(2t) \sin(3t) + 2\exp(2t) \cos(3t) \end{bmatrix}.$$ 

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**Problem 2.**

In each part, you are given a nilpotent matrix $N$. Verify that $N$ is nilpotent and compute $e^{tN}$.

A. 

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

*Answer:*

You can check that $N^2 \neq 0$ but $N^3 = 0$.

Recall that is $N$ is an $n \times n$ nilpotent matrix, then

$$e^{tN} = I + tN + \frac{1}{2!}t^2 N^2 + \frac{1}{3!}t^3 N^2 + \cdots + \frac{1}{(n-1)!}t^{n-1} N^{n-1}.$$
Applying this formula to our particular \( N \), we get
\[
e^{tN} = \begin{bmatrix}
1 & t & (1/2)t^2 \\
0 & 1 & t \\
0 & 0 & 1
\end{bmatrix}.
\]

B.
\[
N = \begin{bmatrix}
537 & 257 & -3656 \\
215 & 103 & -1464 \\
94 & 45 & -640
\end{bmatrix}
\]

Answer:
You can check that \( N^2 \neq 0 \) but \( N^3 = 0 \). The answer is
\[
e^{tN} = \begin{bmatrix}
1 + 537t - 20t^2 & 257t - 20t^2 & -3656t + 160t^2 \\
215t - 8t^2 & 1 + 103t - 8t^2 & -1464t + 64t^2 \\
94t - (7/2)t^2 & 45t - (7/2)t^2 & 1 - 640t + 28t^2
\end{bmatrix}.
\]

C.
\[
N = \begin{bmatrix}
336 & 21 & 84 & 84 \\
-1584 & -20 & 928 & -416 \\
79 & 0 & -63 & 21 \\
-1027 & -79 & -505 & -253
\end{bmatrix}
\]

Answer:
In this case \( N^2 = 0 \). The answer is
\[
e^{tN} = \begin{bmatrix}
1 + 336t & 21t & 84t & 84t \\
-1584t & 1 - 20t & 928t & -416t \\
79t & 0 & 1 - 63t & 21t \\
-1027t & -79t & -505t & 1 - 253t
\end{bmatrix}.
\]

Problem 3.
In each part, you are given a matrix \( A \) which is (definitely) nondiagonalizable. Show that the matrix is nondiagonalizable and find the Jordan Decomposition \( A = S + N \). Use the Jordan Decomposition to compute \( e^{tA} \). If an initial value problem is given, solve the initial value problem.

A.
\[
A = \begin{bmatrix}
323 & -468 & 1140 \\
25 & -37 & 88 \\
-81 & 117 & -286
\end{bmatrix}
\]
Solve the initial value problem
\[ x'(t) = Ax(t) \]
\[ x(0) = c = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \]

**Answer:**
The eigenvalues are 2 and \(-1\).
For eigenvalue 2, we compute
\[ A - 2I = \begin{bmatrix} 321 & -468 & 1140 \\ 25 & -39 & 88 \\ -81 & 117 & -288 \end{bmatrix} . \]
To find the generalized eigenspace \( G(2) = \text{nullspace}(A - 2I)^3 \), we calculate
\[ (A - 2I)^3 = \begin{bmatrix} 3105 & -4212 & 11124 \\ 243 & -351 & 864 \\ -783 & 1053 & -2808 \end{bmatrix} . \]
The RREF of \( (A - 2I)^3 \) is
\[ R = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4/13 \\ 0 & 0 & 0 \end{bmatrix} . \]
From this, we calculate that \( G(2) \) is one dimensional with basis vector
\[ p_1 = \begin{bmatrix} -4 \\ -4/13 \\ 1 \end{bmatrix} . \]
We must have \( G(2) = E(2) \) in this case (why?).
Next consider the eigenvalue \(-1\). We then have
\[ A + I = \begin{bmatrix} 324 & -468 & 1140 \\ 25 & -36 & 88 \\ -81 & 117 & -285 \end{bmatrix} . \]
The RREF of \( A + I \) is
\[ R = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} . \]
From this we see that \( E(-1) \) is one dimensional with basis vector
\[ \begin{bmatrix} -4 \\ -1/3 \\ 1 \end{bmatrix} . \]
Since the dimensions of the eigenspaces don’t add up to 3, we see that $A$ is not diagonalizable.

To find a basis of $G(-1)$, we compute

$$(A + I)^3 = \begin{bmatrix} 2808 & -4212 & 9828 \\ 216 & -324 & 756 \\ -702 & 1053 & -2457 \end{bmatrix}.$$ 

The RREF of $(A + I)^3$ is

$$R = \begin{bmatrix} 1 & -3/2 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Computing the nullspace of $R$, we conclude that $G(-1)$ is two dimensional with basis

$$p_2 = \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} -7/2 \\ 0 \\ 1 \end{bmatrix}.$$ 

Next, we put our basis vectors for the generalized eigenspaces together to get a matrix $P$,

$$P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -4 & 3/2 & -7/2 \\ -4/13 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$ 

and we construct the diagonal matrix

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where the entry on the main diagonal in each column is the eigenvalue that goes with the corresponding column of $P$. We can then calculate

$$S = PDP^{-1} = \begin{bmatrix} 311 & -468 & 1092 \\ 24 & -37 & 84 \\ -78 & 117 & -274 \end{bmatrix}.$$ 

You might use your calculator to check that $AS = SA$.

We can then calculate the nilpotent part $N$ of the Jordan Decomposition by

$$N = A - S = \begin{bmatrix} 12 & 0 & 48 \\ 1 & 0 & 4 \\ -3 & 0 & -12 \end{bmatrix}.$$ 

You can use your calculator to check that $N^2 = 0$. 

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To calculate $e^{tS}$ we first calculate

$$e^{tD} = \begin{bmatrix}
\exp(2t) & 0 & 0 \\
0 & \exp(-t) & 0 \\
0 & 0 & \exp(-t)
\end{bmatrix}.$$  

We then have

$$e^{tS} = Pe^{tD}P^{-1} = \begin{bmatrix}
104 \exp(2t) - 103 \exp(-t) & -156 \exp(2t) + 156 \exp(-t) & 364 \exp(2t) - 364 \exp(-t) \\
8 \exp(2t) - 8 \exp(-t) & -12 \exp(2t) + 13 \exp(-t) & 28 \exp(2t) - 28 \exp(-t) \\
-26 \exp(2t) + 26 \exp(-t) & 39 \exp(2t) - 39 \exp(-t) & -91 \exp(2t) + 92 \exp(-t)
\end{bmatrix}. $$

Since $N$ is nilpotent, we can calculate $e^{tN}$ (as in a previous problem). We get

$$e^{tN} = \begin{bmatrix}
1 + 12t & 0 & 48t \\
t & 1 & 4t \\
-3t & 0 & 1 - 12t
\end{bmatrix}.$$  

Since $S$ and $N$ commute, we have

$$e^{tA} = e^{tS}e^{tN} = \begin{bmatrix}
104 \exp(2t) - 103 \exp(-t) + 12t \exp(-t) & -156 \exp(2t) + 156 \exp(-t) + 48t \exp(-t) + 364 \exp(2t) - 364 \exp(-t) \\
8 \exp(2t) - 8 \exp(-t) + t \exp(-t) & -12 \exp(2t) + 13 \exp(-t) + 4t \exp(-t) + 28 \exp(2t) - 28 \exp(-t) \\
-26 \exp(2t) + 26 \exp(-t) - 3t \exp(-t) & 39 \exp(2t) - 39 \exp(-t) - 12t \exp(-t) + 91 \exp(2t) + 92 \exp(-t)
\end{bmatrix}. $$

I had to shrink that to get it on the page. I did some simplification to get that, your answer may look different but should be equivalent.

You can check your answer by using the calculator to compute

$$\frac{d}{dt} e^{tA} - Ae^{tA}$$

which should give zero. Also check that the value at $t = 0$ is $I$.

The solution to the given initial value problem is

$$x(t) = \begin{bmatrix}
468 \exp(2t) - 469 \exp(-t) + 84t \exp(-t) \\
36 \exp(2t) - 35 \exp(-t) + 7t \exp(-t) \\
-117 \exp(2t) + 119 \exp(-t) - 21t \exp(-t)
\end{bmatrix}$$

B.

$$A = \begin{bmatrix}
25 & 38 & 2 & -12 \\
12 & 20 & 1 & -6 \\
-2 & -3 & 2 & 1 \\
82 & 130 & 7 & -40
\end{bmatrix}.$$  

Answer:

The eigenvalues are 1 and 2.
The generalized eigenspace $G(1)$ is one dimensional with basis vector

$$p_1 = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

For eigenvalue 2, we have

$$A - 2I = \begin{bmatrix} 23 & 38 & 2 & -12 \\ 12 & 18 & 1 & -6 \\ -2 & -3 & 0 & 1 \\ 82 & 130 & 7 & -42 \end{bmatrix}.$$

The nullspace of $A - 2I$, which is $E(2)$, is one dimensional with basis vector

$$u = \begin{bmatrix} 2/7 \\ 1/7 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, $A$ is not diagonalizable.

To find a basis $G(2) = \text{nullspace}((A - 2I)^4)$, we calculate

$$(A - 2I)^4 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 \end{bmatrix}.$$

The RREF of $(A - 2I)^4$ is

$$R = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Calculating the nullspace of $R$, we find that $G(2)$ is three dimensional with basis vectors

$$p_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(Remark: The eigenvector $u$ we found above should be in the span of these vectors. Is it?)
We then form the matrices

\[ P = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \]

and

\[ D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \]

We can then calculate \( S \) and \( N \) as

\[ S = PDP^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -2 & 4 & 0 & 2 \end{bmatrix}, \]

\[ N = A - S = \begin{bmatrix} 24 & 36 & -12 \\ 12 & 18 & -6 \\ -2 & -3 & 0 & 1 \\ 84 & 126 & 7 & -42 \end{bmatrix}. \]

You might use your calculator to check that \( AS - SA = 0; N^2 \neq 0 \) but \( N^3 = 0 \).

To calculate \( e^{tS} \) we have

\[ e^{tD} = \begin{bmatrix} \exp(t) & 0 & 0 & 0 \\ 0 & \exp(2t) & 0 & 0 \\ 0 & 0 & \exp(2t) & 0 \\ 0 & 0 & 0 & \exp(2t) \end{bmatrix}, \]

\[ e^{tS} = Pe^{tD}P^{-1} = \begin{bmatrix} \exp(t) & -2\exp(t) + 2\exp(2t) & 0 & 0 \\ 0 & \exp(2t) & 0 & 0 \\ 0 & 0 & \exp(2t) & 0 \\ 2\exp(t) - 2\exp(2t) & -4\exp(t) + 4\exp(2t) & 0 & \exp(2t) \end{bmatrix}. \]

By the usual method,

\[ e^{tN} = \begin{bmatrix} 1 + 24t - 2t^2 & 36t - 3t^2 & 2t & -12t + t^2 \\ 12t - t^2 & 1 + 18t - \frac{3}{2}t^2 & t & -6t + \frac{1}{2}t^2 \\ -2t & -3t & 1 & t \\ 84t - 7t^2 & 126t - \frac{21}{2}t^2 & 7t & 1 - 42t + \frac{7}{2}t^2 \end{bmatrix}. \]

We can then compute \( e^{tA} = e^{tN}e^{tS} \). After trying to make it look nice, I get the following (sorry, wouldn’t fit on the page without rotating it).
\[ e^{tA} = \begin{bmatrix}
\exp(t) + (-2t^2 + 24t) \exp(2t) & -2 \exp(t) + (-3t^2 + 36t + 2) \exp(2t) & 2t \exp(2t) & t(t - 12) \exp(2t) \\
-t(t - 12) \exp(2t) & (-\frac{3}{2}t^2 + 18t + 1) \exp(2t) & t \exp(2t) & \frac{1}{2}t(t - 12) \exp(2t) \\
-2t \exp(2t) & -3t \exp(2t) & \exp(2t) & t \exp(2t) \\
2 \exp(t) + (-7t^2 + 84t - 2) \exp(2t) & -4 \exp(t) + (-\frac{21}{2}t^2 + 126t + 4) \exp(2t) & 7t \exp(2t) & \left(\frac{7}{2}t^2 - 42t + 1\right) \exp(2t)
\end{bmatrix}. \]
Try checking your answer, as in the previous part of the problem.

C. 

\[ A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix} \]

**Answer:**

The eigenvalues are \(1 + 2i\) and \(1 - 2i\).

It turns out the eigenspaces \(E(1+2i)\) and \(E(1-2i)\) are both one dimensional, so \(A\) is not diagonalizable.

To compute the generalized eigenspace \(G(1 + 2i)\), we compute

\[ (A - (1 + 2i)I)^4 = \begin{bmatrix} 128 & -128i & 128i & 128 \\ 128i & 128 & -128 & 128i \\ 0 & 0 & 128 & -128i \\ 0 & 0 & 128i & 128 \end{bmatrix} \]

The RREF of \((A - (1 + 2i)I)^4\) is

\[ R = \begin{bmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

Computing the nullspace of \(R\), we find that \(G(1 + 2i)\) is two dimensional with basis

\[ p_1 = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}. \]

To get a basis for \(G(1 - 2i)\) we can just take the conjugates of the basis we got for \(G(1 + 2i)\). Thus, our basis of \(G(1 - 2i)\) is

\[ p_3 = \begin{bmatrix} 0 \\ 0 \\ -i \\ 1 \end{bmatrix}, \quad p_4 = \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}. \]
We then get

\[
P = \begin{bmatrix} 0 & i & 0 & -i \\ 0 & 1 & 0 & 1 \\ i & 0 & -i & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}
\]

\[
D = \begin{bmatrix} 1 + 2i & 0 & 0 & 0 \\ 0 & 1 + 2i & 0 & 0 \\ 0 & 0 & 1 - 2i & 0 \\ 0 & 0 & 0 & 1 - 2i \end{bmatrix}
\]

\[
S = PDP^{-1} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix}
\]

\[
N = A - S = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

To compute \(e^{tS}\) we compute

\[
e^{tD} = \begin{bmatrix} e^{(1+2i)t} & 0 & 0 & 0 \\ 0 & e^{(1+2i)t} & 0 & 0 \\ 0 & 0 & e^{(1-2i)t} & 0 \\ 0 & 0 & 0 & e^{(1-2i)t} \end{bmatrix}
\]

\[
e^{tS} = Pe^{tD}P^{-1} = \begin{bmatrix} \exp(t) \cos(2t) & -\exp(t) \sin(2t) & 0 & 0 \\ \exp(t) \sin(2t) & \exp(t) \cos(2t) & 0 & 0 \\ 0 & 0 & \exp(t) \sin(2t) & \exp(t) \cos(2t) \\ 0 & 0 & 0 & \exp(t) \cos(2t) \end{bmatrix}.
\]

In this case \(N^2 = 0\), so

\[
e^{tN} = I + tN = \begin{bmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

Putting these together, \(e^{tA} = e^{tN}e^{tS}\), so

\[
e^{tA} = \begin{bmatrix} \exp(t) \cos(2t) & -\exp(t) \sin(2t) & t \exp(t) \cos(2t) & -t \exp(t) \sin(2t) \\ \exp(t) \sin(2t) & \exp(t) \cos(2t) & t \exp(t) \sin(2t) & t \exp(t) \cos(2t) \\ 0 & 0 & \exp(t) \cos(2t) & -\exp(t) \sin(2t) \\ 0 & 0 & 0 & \exp(t) \cos(2t) \end{bmatrix}.
\]