EXAM

Exam 2

Math 2360, Spring 2010

April 6, 2011

• Write all of your answers on separate sheets of paper. You can keep the exam questions when you leave. You may leave when finished.

• You must show enough work to justify your answers.

• The use of a TI-89 Calculator is expected. State clearly which computations you are doing on the calculator. In all cases, give exact answers and express your numbers as integers or fractions, not in decimal form (1/3 good, 0.3333 bad).

• This exam has 5 problems. There are 320 points total.

Good luck!
Problem 1. Consider the matrix

\[ A = \begin{bmatrix}
1 & 1 & 1 & 3 & -10 \\
-2 & -1 & 1 & 0 & 3 \\
4 & 2 & -2 & 3 & -12 \\
-4 & -2 & 2 & 0 & 6 \\
-1 & -1 & -1 & -4 & 12
\end{bmatrix}. \]

The RREF of \( A \) is the matrix

\[ R = \begin{bmatrix}
1 & 0 & -2 & 0 & 1 \\
0 & 1 & 3 & 0 & -5 \\
0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}. \]

A. Find a basis for the nullspace of \( A \).
B. Find a basis for the rowspace of \( A \).
C. Find a basis for the columnspace of \( A \).
D. What is the rank of \( A \)?

Problem 2. Let \( A \) be be a 5 × 7 matrix.

A. What is the maximum possible value of the rank of \( A \)?
B. If the rank of \( A \) is 4, what is the dimension of the nullspace of \( A \)?

Problem 3. You’ll want a calculator for this problem. Consider the vectors

\[ v_1 = \begin{bmatrix} 4 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 18 \\ 10 \\ 1 \\ 4 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \]

Let \( S = \text{span}(v_1, v_2, v_3, v_4) \), which is a subspace of \( \mathbb{R}^4 \).

A. Find a basis of \( S \). What is the dimension of \( S \)?
B. Express the vectors in the list $v_1, \ldots, v_4$ that are not part of the basis you found as linear combinations of the basis vectors.

C. Consider the vectors

$$w_1 = \begin{bmatrix} 8 \\ 7 \\ 4 \\ 6 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -5 \\ 7 \\ 2 \\ 3 \end{bmatrix}.$$  

Determine if these vectors are in $S$. If the vector is in $S$, express it as a linear combination of the basis vectors found above.

---

**Problem 4.** Recall that the standard basis of $\mathbb{R}^2$ is $E = [e_1 \ e_2]$ where

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$  

Let

$$U = [u_1 \ u_2],$$

where

$$u_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

and let

$$V = [v_1 \ v_2],$$

where

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$  

Then $U$ and $V$ are ordered bases of $\mathbb{R}^2$ (you don’t need to check that).

A. Find the change of basis matrices $S_{EU}$ and $S_{EV}$.

B. Find the change of basis matrices $S_{UV}$ and $S_{VU}$.

C. Let $w \in \mathbb{R}^2$ be the vector such that

$$[w]_U = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$  

Find $[w]_E$ and express $w$ as an element of $\mathbb{R}^2$.

D. Find $[w]_V$, the coordinate vector of $w$ with respect to $V$. 

---

2
Problem 5. Recall that the standard basis of \( \mathbb{R}^2 \) is \( \mathcal{E} = [e_1 \ e_2] \) where
\[
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Let
\[
\mathcal{U} = [u_1 \ u_2],
\]
where
\[
u_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},
\]
and let
\[
\mathcal{V} = [v_1 \ v_2],
\]
where
\[
v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

Then \( \mathcal{U} \) and \( \mathcal{V} \) are ordered bases of \( \mathbb{R}^2 \) (you don’t need to check that). You can use the results of the previous problem without repeating the calculations here.

Let \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation such that
\[
L(u_1) = u_1 - 3u_2
\]
\[
L(u_2) = -2u_1 + 2u_2.
\]

A. Find \( [L]_{\mathcal{U}} \), the matrix of \( L \) with respect to the basis \( \mathcal{U} \).

B. Find \( [L]_{\mathcal{V}} \), the matrix of \( L \) with respect to the basis \( \mathcal{V} \).

C. Let \( w \in \mathbb{R}^2 \) be the vector whose coordinate vector with respect to \( \mathcal{V} \) is
\[
[w]_{\mathcal{V}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\]

Find \( [L(w)]_{\mathcal{V}} \), the coordinate vector of \( L(v) \) with respect to the basis \( \mathcal{V} \).