PROBLEM SET

Problems on Span, Independence, and Matrix Spaces

Math 3351, Fall 2010

Sept. 27, 2010

- Write all of your answers on separate sheets of paper. You can keep the question sheet.
- You must show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This problem set has 8 problems.

Good luck!
Problem 1. Consider the matrix

\[
A = \begin{bmatrix}
-15 & -80 & -145 & 22 & -181 \\
-89 & -19 & 51 & 50 & 132 \\
66 & -62 & -190 & 78 & -96 \\
77 & 81 & 85 & -8 & 150
\end{bmatrix}.
\]

A Find a basis of the nullspace of \( A \).
B Find a basis of the rowspace of \( A \).
C Find a basis of the column space of \( A \). Write the other columns of \( A \) as linear combinations of these basis vectors.
D What is the rank of \( A \)?

Problem 2. Consider the matrix

\[
A = \begin{bmatrix}
8 & 9 & 19 & 4 & 22 & -25 \\
7 & 1 & -4 & 6 & 3 & 10 \\
5 & 5 & 10 & 2 & 13 & -12 \\
2 & 7 & 19 & 4 & 12 & -33 \\
4 & 4 & 8 & 6 & 6 & -14 \\
4 & 1 & -1 & 6 & 0 & 1
\end{bmatrix}
\]

A Find a basis of the nullspace of \( A \).
B Find a basis of the rowspace of \( A \).
C Find a basis of the column space of \( A \). Write the other columns of \( A \) as linear combinations of these basis vectors.
D What is the rank of \( A \)?
Problem 3. Consider the vectors

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} 5 \\ 6 \\ 2 \\ 0 \end{bmatrix}, & \mathbf{v}_2 &= \begin{bmatrix} 7 \\ 8 \\ 1 \\ 4 \end{bmatrix}, & \mathbf{v}_3 &= \begin{bmatrix} 11 \\ 12 \\ -1 \\ 12 \end{bmatrix}, & \mathbf{v}_4 &= \begin{bmatrix} 14 \\ 3 \\ 3 \\ 4 \end{bmatrix}, & \mathbf{v}_5 &= \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}.
\end{align*}
\]

Let \( S = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5) \), which is a subspace of \( \mathbb{R}^4 \).

A. Find a basis of \( S \). What is the dimension of \( S \)?

B. Express the vectors in the list \( \mathbf{v}_1, \ldots, \mathbf{v}_5 \) that are not part of the basis as linear combinations of the basis vectors.

C. Consider the vectors

\[
\begin{align*}
\mathbf{w}_1 &= \begin{bmatrix} 7 \\ 9 \\ 0 \\ 3 \end{bmatrix}, & \mathbf{w}_2 &= \begin{bmatrix} 12 \\ 13 \\ 27 \\ -1 \end{bmatrix}.
\end{align*}
\]

Determine if these vectors are in \( S \). If the vector is in \( S \), express it as a linear combination of the basis vectors found above.

Problem 4. Consider the vectors

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} 5 \\ 6 \\ 2 \\ 0 \end{bmatrix}, & \mathbf{v}_2 &= \begin{bmatrix} 7 \\ 8 \\ 1 \\ 4 \end{bmatrix}, & \mathbf{v}_3 &= \begin{bmatrix} 11 \\ 12 \\ -1 \\ 12 \end{bmatrix}, & \mathbf{v}_4 &= \begin{bmatrix} 14 \\ 3 \\ 3 \\ 4 \end{bmatrix}, & \mathbf{v}_5 &= \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}.
\end{align*}
\]

Let \( S = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5) \), which is a subspace of \( \mathbb{R}^5 \).

A. Find a basis of \( S \). What is the dimension of \( S \)?

B. Express the vectors in the list \( \mathbf{v}_1, \ldots, \mathbf{v}_5 \) that are not part of the basis as linear combinations of the basis vectors.
C. Consider the vectors

\[ w_1 = \begin{bmatrix} 7 \\ 9 \\ 0 \\ 3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \end{bmatrix} \]

Determine if these vectors are in \( S \). If the vector is in \( S \), express it as a linear combination of the basis vectors found above.

**Problem 5.** Determine if the following vectors are independent or dependent. If they are independent, complete the list to a basis of \( \mathbb{R}^5 \) by adding on a standard basis vector. If the vectors are dependent, find a linear relation between them.

\[ v_1 = \begin{bmatrix} 13 \\ -10 \\ 24 \\ -9 \\ -11 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ -1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ -4 \end{bmatrix} \]

**Problem 6.** Determine if the following vectors are independent or dependent. If they are independent, complete the list to a basis of \( \mathbb{R}^5 \) by adding on a standard basis vector. If the vectors are dependent, find a linear relation between them.
relation between them.

\[
\begin{align*}
v_1 &= \begin{bmatrix} -38 \\ 91 \\ -1 \\ 63 \\ -23 \end{bmatrix}, \\
v_2 &= \begin{bmatrix} -63 \\ -26 \\ 30 \\ 10 \\ 22 \end{bmatrix}, \\
v_3 &= \begin{bmatrix} 12 \\ 45 \\ -14 \\ 60 \\ -35 \end{bmatrix}, \\
v_4 &= \begin{bmatrix} -89 \\ -170 \\ 64 \\ 24 \\ 42 \end{bmatrix}
\end{align*}
\]

**Problem 7.** Consider the vectors

\[
\begin{align*}
v_1 &= \begin{bmatrix} 4 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \\
v_2 &= \begin{bmatrix} 4 \\ 5 \\ 3 \\ 1 \end{bmatrix}, \\
v_3 &= \begin{bmatrix} 4 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \\
v_4 &= \begin{bmatrix} 28 \\ 26 \\ 12 \\ 7 \end{bmatrix}, \\
v_5 &= \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix}, \\
v_6 &= \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}
\end{align*}
\]

Do these vectors span \( \mathbb{R}^4 \)? If so, select a basis of \( \mathbb{R}^4 \) out of this list of vectors.
Problem 8. Consider the vectors

\[ v_1 = \begin{bmatrix} 3 \\ 6 \\ 5 \\ 8 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 9 \\ 4 \\ 2 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ -8 \\ -8 \\ -12 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 3 \\ 0 \\ 7 \\ 2 \end{bmatrix}, \]

\[ v_5 = \begin{bmatrix} 9 \\ 10 \\ 0 \\ 10 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 3 \\ 4 \\ 8 \\ 0 \end{bmatrix}, \quad v_7 = \begin{bmatrix} 42 \\ 22 \\ 47 \\ 8 \end{bmatrix} \]

Do these vectors span \( \mathbb{R}^4 \)? If so, select a basis of \( \mathbb{R}^4 \) out of this list of vectors.