EXAM

Practice Problems for Exam 2

Math 4350, Fall 2009

November 24, 2009

• These are practice problems to give you an idea of what will be on the exam. Of course, there are more problems here than I could put on an inclass exam.

• This exam has 13 problems.

Good luck!
Problem 1. Let $A$ be a subset of $\mathbb{R}$. Define what it means for a point $p \in \mathbb{R}$ to be a **cluster point** of $A$.

Problem 2. Let $A$ be a subset of $\mathbb{R}$, let $f : A \rightarrow \mathbb{R}$ be a function and let $p$ be a cluster point of $A$. Give the formal definition of

$$
\lim_{x \to p} f(x) = L,
$$

where $L$ is a real number, i.e., give the $\varepsilon\delta$ definition.

Problem 3. Let $f : \mathbb{R} \supseteq A \rightarrow \mathbb{R}$ be a function. Give the formal definition of the following concepts. State any necessary assumptions on the domain $A$ for the definition to make sense.

A. 
\[ 
\lim_{x \to \infty} f(x) = L, \quad L \in \mathbb{R} 
\]

B. 
\[ 
\lim_{x \to p} f(x) = \infty, \quad \text{where } p \in \mathbb{R}. 
\]

Problem 4. Give a restriction on $|x - 2|$ that implies $|x^3 - 8| < 1/100$. Hint: 
\[ x^3 - 8 = (x - 2)(x^2 + 2x + 4). \]

Problem 5. Let $A \subseteq \mathbb{R}$. Let $p$ be a cluster point of $A$ and let $f, g$ and $h$ be functions $A \rightarrow \mathbb{R}$. Suppose that 
\[ f(x) \leq g(x) \leq h(x) \]
for all $x \in A$ and that 
\[ 
\lim_{x \to p} f(x) = \lim_{x \to p} h(x) = L 
\]
for some $L \in \mathbb{R}$. Show that 
\[ 
\lim_{x \to p} g(x) = L. 
\]
(This is the “squeeze theorem”.)
Problem 6. Let $f : \mathbb{R} \subseteq A \to \mathbb{R}$. Let $p$ be a cluster point of $A$ and that
\[
\lim_{x \to p} f(x) = L, \quad L \in \mathbb{R}, \: L \neq 0.
\]
Show that
\[
\lim_{x \to p} \frac{1}{f(x)}
\]
makes sense and
\[
\lim_{x \to p} \frac{1}{f(x)} = \frac{1}{L}.
\]

Problem 7. Let $g : \mathbb{R} \supseteq A \to \mathbb{R}$ be a function. Suppose that $p$ is a cluster point of $A$ and that
\[
\lim_{x \to p} g(x) = L, \quad L \in \mathbb{R}.
\]
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at $L$. Show that
\[
\lim_{x \to p} f(g(x)) = f(L).
\]

Problem 8. Let $f : \mathbb{R} \supseteq A \to \mathbb{R}$. Let $p$ be a point of $A$.
Show that the following conditions are equivalent.
A. $f$ is continuous at $p$.
B. If $(x_n)$ is a sequence in $A$ that converges to $p$, then
\[
\lim(f(x_n)) = f(p).
\]

Problem 9. Let $f : \mathbb{R} \supseteq A \to \mathbb{R}$ be continuous at a point $p \in A$. Suppose that $f(p) > 0$. Show there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in V_\delta(p) \cap A$.

Problem 10. Let $f$ be continuous on $[a, b]$. Show that $f$ is bounded on $[a, b]$.

Problem 11. Let $f$ be a continuous function on $[0, \infty)$ and suppose
\[
\lim_{x \to \infty} f(x) = 0.
\]
Show that $f$ assumes either a maximum or a minimum on $[0, \infty)$. Give an example to show that $f$ can have a maximum but no minimum, or vice-versa.
Problem 12. Let \( f : \mathbb{R} \supseteq A \to \mathbb{R} \). Define what it means for \( f \) to be uniformly continuous on \( A \). (an \( \varepsilon \delta \)-definition).

Problem 13. Let \( f \) and \( g \) be uniformly continuous functions \( \mathbb{R} \to \mathbb{R} \). Show that \( f \circ g \) is uniformly continuous.