This is a Take Home Exam. It is due on Wednesday, August 8, by Noon.

You must show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).

You may discuss the problems with other people, but write up the solutions by yourself.

You will have to use a calculator. In particular, you can use the calculator to do matrix algebra, dot products, and find the RREF of a matrix. Say what you are computing with the calculator and give the result. If there are any questions on when it is legal to use a calculator, ask me.

This exam has 8 problems. There are 410 points total.

Good luck!
Problem 1. The matrix

\[ A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \]

is invertible. Use row operations to express \( A \) as a product of elementary matrices. You can use a calculator to do the row operations, but you’ll have to show each row operation.

Problem 2. Consider the vectors

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \\
\mathbf{v}_2 &= \begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \\
\mathbf{v}_3 &= \begin{bmatrix} -2 \\ -3 \\ 0 \\ -3 \end{bmatrix}, \\
\mathbf{v}_4 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \\
\mathbf{v}_5 &= \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}.
\end{align*}
\]

Let \( S \subset \mathbb{R}^5 \) be defined by

\[ S = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5). \]

A. Find a basis for \( S \). What is the dimension of \( S \)?

B. Consider the vectors

\[
\begin{align*}
\mathbf{w}_1 &= \begin{bmatrix} 9 \\ 6 \\ 1 \\ 8 \end{bmatrix}, \\
\mathbf{w}_2 &= \begin{bmatrix} 3 \\ -1 \\ -1 \\ 2 \end{bmatrix}.
\end{align*}
\]

Determine if each of these vectors is in \( S \). If the vector is in \( S \), write it as a linear combination of the basis vectors for \( S \) you found in the first part.

Problem 3. In this problem, we’re working in the vector space

\[ P_3 = \{ ax^2 + bx + c \mid a, b, c \in \mathbb{R} \}, \]

the space of polynomials of degree less than three. Let \( \mathcal{U} \) be the basis of \( P_3 \) given by

\[ \mathcal{U} = [x^2 \ x \ 1]. \]

Let \( T : P_3 \to P_3 \) be the linear transformation defined by

\[ T(p(x)) = p'(x) + 2p(x). \]

Find \( [T]_{\mathcal{U}\mathcal{U}} \), the matrix of \( T \) with respect to the basis \( \mathcal{U} \).
Problem 4. Let \( \mathcal{U} = [u_1 \ u_2] \) be the basis of \( \mathbb{R}^2 \), where
\[
\begin{align*}
    u_1 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \\
    u_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{align*}
\]

A. Find the change of basis matrices \( S_{\mathcal{E}\mathcal{U}} \) and \( S_{\mathcal{U}\mathcal{E}} \), where \( \mathcal{E} \) is the standard basis of \( \mathbb{R}^2 \).

B. If \( v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), find \([v]_{\mathcal{U}}\), the coordinates of \( v \) with respect to \( \mathcal{U} \).

C. If \([w]_{\mathcal{U}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\), find \( w \).

D. Let \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the linear transformation that satisfies
\[
\begin{align*}
    T(u_1) &= 2u_1 + 3u_2, \\
    T(u_2) &= u_1 - u_2.
\end{align*}
\]

Find \([T]_{\mathcal{U}\mathcal{U}}\), the matrix of \( T \) with respect to \( \mathcal{U} \), and \([T]_{\mathcal{E}\mathcal{E}}\), the matrix of \( T \) with respect to \( \mathcal{E} \).

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Problem 5. Let
\[
    A = \begin{bmatrix} -4 & 3 \\ -2 & 3 \end{bmatrix}
\]

Find the characteristic polynomial and the eigenvalues of \( A \). (Do not find any eigenvectors.)
Problem 6. In each part, you are given a matrix $A$ and its eigenvalues. Find a basis for each of the eigenspaces of $A$ and determine if $A$ is diagonalizable. If so, find a diagonal matrix $D$ and an invertible matrix $P$ so that $P^{-1}AP = D$.

A. The matrix is

$$A = \begin{bmatrix} 11 & -3 & -3 \\ 9 & -1 & -3 \\ 27 & -9 & -7 \end{bmatrix}$$

and the eigenvalues are $-1$ and $2$.

B. The matrix is

$$A = \begin{bmatrix} -4 & -1 & 4 \\ -6 & -2 & 7 \\ -6 & -1 & 6 \end{bmatrix}$$

and the eigenvalues are $-1$ and $2$.

C. The matrix is

$$A = \begin{bmatrix} -3 & 2 & 2 \\ -1 & 5 & -4 \\ -2 & 2 & 1 \end{bmatrix}$$

and the eigenvalues are $-1$, $2 + i$ and $2 - i$.

Problem 7. Consider the three vectors

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$  

Apply the Gram-Schmidt Process to these vectors to produce an orthonormal basis of $\mathbb{R}^3$.  

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Problem 8. Let $S$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$  

A. Find an orthonormal basis for $S$.

B. Find an orthonormal basis for $S^\perp$.

C. Determine if each of the following vectors is in $S$ by computing inner products.

$$w_1 = \begin{bmatrix} -1 \\ -1 \\ -3 \\ 2 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 4 \end{bmatrix}.$$  