Define a function \( f : \mathbb{N} \rightarrow \mathbb{N} \) by
\[
    f(n) = \begin{cases} 
        3n + 1, & \text{if } n \text{ is odd}, \\
        n/2, & \text{if } n \text{ is even}.
    \end{cases}
\]

Use the notation \( f^k \) to denote the composition of \( f \) with itself \( k \) times, so that \( f^2(n) = f(f(n)) \), \( f^3(n) = f(f(f(n))) \), and so on. There is a famous conjecture that for each positive integer \( n \), the sequence \( n, f(n), f^2(n), f^3(n), \ldots \) eventually reaches 1. For example, if \( n = 6 \) this sequence is 6, 3, 10, 5, 16, 8, 4, 2, 1, \ldots.

For each positive integer \( n \), let \( \lambda_n \) be the least positive integer such that \( f^{\lambda_n}(n) = 1 \). From the example above, \( \lambda_6 = 8 \).

**Extra credit (10 points)**

- Write Python code to determine the average of \( \lambda_1, \lambda_2, \ldots, \lambda_{10000} \).

The following will do it:

```python
def f(n):
    if n%2==0:
        return n/2
    return 3*n+1

def lam(n):
    k=0
    N=n
    while N != 1:
        N = f(N)
        k += 1
    return k

def mean_lam(x):
    total = 0
    n = 1
    while n <= x:
        total += lam(n)
        n += 1
    return float(total)/x
```

---

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print mean_lam(10000)

The average of $\lambda_1, \ldots, \lambda_{10000}$ is about 84.97.

• Tweak your code to determine the average of $\lambda_1, \lambda_2, \ldots, \lambda_{10000}$. (You may need to determine several more such averages, so consider writing a function which takes a parameter $t$ and returns the average of $\lambda_1, \lambda_2, \ldots, \lambda_t$).

This average is about 107.54

• Make a conjecture about the average value of $\lambda_1, \lambda_2, \ldots, \lambda_x$. (You may need to compute several more averages to make a reasonable conjecture, so feel free to do so).

After doing several similar calculations, we find

<table>
<thead>
<tr>
<th>$t$</th>
<th>Average of $\lambda_1, \ldots, \lambda_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^4$</td>
<td>59.54</td>
</tr>
<tr>
<td>$10^4$</td>
<td>84.97</td>
</tr>
<tr>
<td>$10^5$</td>
<td>107.54</td>
</tr>
<tr>
<td>$10^6$</td>
<td>131.43</td>
</tr>
<tr>
<td>$10^7$</td>
<td>155.27</td>
</tr>
</tbody>
</table>

From here, any number of reasonable conjectures (such as logarithmic growth) are possible. One can also work out some conjectures using probabilistic arguments. For example, if we conjecture logarithmic growth, we might add one more column to the table above to check:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\alpha_t = \text{Average of } \lambda_1, \ldots, \lambda_t$</th>
<th>$\alpha_t / \log t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^4$</td>
<td>59.54</td>
<td>8.62</td>
</tr>
<tr>
<td>$10^4$</td>
<td>84.97</td>
<td>9.23</td>
</tr>
<tr>
<td>$10^5$</td>
<td>107.54</td>
<td>9.34</td>
</tr>
<tr>
<td>$10^6$</td>
<td>131.43</td>
<td>9.51</td>
</tr>
<tr>
<td>$10^7$</td>
<td>155.27</td>
<td>9.63</td>
</tr>
<tr>
<td>$10^8$</td>
<td>179.23</td>
<td>9.73</td>
</tr>
</tbody>
</table>

Here, there’s not quite enough evidence to suggest purely logarithmic growth - it might be growing just a touch faster, but it’s a reasonable first guess.