EXAM

Exam 3
Takehome Exam

Math 2360–D01, Spring 2015

April 23, 2015

• Write all of your answers on separate sheets of paper. Do not write on the exam handout. You can keep the exam questions when you leave. You may leave when finished.

• You must show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).

• This exam has 7 problems. There are 340 points total.

Good luck!
Problem 1. Consider the space $P_3$ of polynomials of degree less than 3. Two ordered bases of this space are
\[
P = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \\
Q = \begin{bmatrix} 1 + 2x + 2x^2 & 2 + x + 2x^2 & 1 + x^2 \end{bmatrix}.
\]
A. Find the change of basis matrices $S_{PQ}$ and $S_{QP}$.
B. Let $f(x) = 2 + x + x^2$. Find $[f(x)]_P$, the coordinate vector of $f(x)$ with respect to $P$. Find $[f(x)]_Q$, the coordinate vector of $f(x)$ with respect to $Q$. Use this information to write $f(x)$ as a linear combination of the entries of $Q$.

Problem 2. Recall that
\[
P = \begin{bmatrix} 1 & x & x^2 \end{bmatrix}
\]
is an ordered basis of $P_3$.
Another ordered basis for $P_3$ is
\[
Q = \begin{bmatrix} 2x^2 - x + 3 & x^2 - 1 & 3x^2 - 2x + 2 \end{bmatrix}
\]
Let $T: P_3 \rightarrow P_3$ be the linear transformation defined by
\[
T(p(x)) = p'(x) + 2p(x).
\]
If it’s not obvious to you that this is linear, check it.
A. Find the matrix of $T$ with respect to the basis $P$, i.e., find $[T]_P$. 
B. Find the matrix of $T$ with respect to the basis $Q$, i.e., find $[T]_Q$.
C. Let $g(x)$ be the element of $P_3$ with $[g(x)]_Q = \begin{bmatrix} -2 & 1 & 3 \end{bmatrix}^T$. Find $[T(g(x))]_Q$. Write $g(x)$ and $T(g(x))$ as linear combinations of $Q$. 

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Problem 3. Recall that the standard basis of $\mathbb{R}^2$ is $\mathcal{E} = [e_1 \ e_2]$ where

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$ 

Let

$$\mathcal{U} = [u_1 \ u_2],$$

where

$$u_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

and let

$$\mathcal{V} = [v_1 \ v_2],$$

where

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ 

Then $\mathcal{U}$ and $\mathcal{V}$ are ordered bases of $\mathbb{R}^2$ (you don’t need to check that).

A. Find the change of basis matrices $S_{\mathcal{E}\mathcal{U}}$ and $S_{\mathcal{E}\mathcal{V}}$.

B. Find the change of basis matrices $S_{\mathcal{U}\mathcal{V}}$ and $S_{\mathcal{V}\mathcal{U}}$.

C. Let $w \in \mathbb{R}^2$ be the vector such that

$$[w]_\mathcal{U} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$ 

Find $[w]_\mathcal{E}$ and express $w$ as column vector in $\mathbb{R}^2$.

D. Find $[w]_\mathcal{V}$, the coordinate vector of $w$ with respect to $\mathcal{V}$. 

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Problem 4. Recall that the standard basis of \( \mathbb{R}^2 \) is \( \mathcal{E} = [e_1 \ e_2] \) where

\[
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Let

\[
\mathcal{U} = [u_1 \ u_2],
\]

where

\[
u_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},
\]

and let

\[
\mathcal{V} = [v_1 \ v_2],
\]

where

\[
v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

Then \( \mathcal{U} \) and \( \mathcal{V} \) are ordered bases of \( \mathbb{R}^2 \) (you don’t need to check that). You can use the results of the previous problem without repeating the calculations here.

Let \( L: \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation such that

\[
L(u_1) = u_1 - 3u_2 \\
L(u_2) = -2u_1 + 2u_2.
\]

A. Find \( [L]_{\mathcal{U}U} \), the matrix of \( L \) with respect to the basis \( \mathcal{U} \).

B. Find \( [L]_{\mathcal{V}V} \), the matrix of \( L \) with respect to the basis \( \mathcal{V} \).

C. Let \( w \in \mathbb{R}^2 \) be the vector whose coordinate vector with respect to \( \mathcal{V} \) is

\[
[w]_{\mathcal{V}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\]

Find \( [L(w)]_{\mathcal{V}} \), the coordinate vector of \( L(w) \) with respect to the basis \( \mathcal{V} \).

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Problem 5. Let

\[
A = \begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}.
\]

Find the characteristic polynomial and the eigenvalues of \( A \). (Do not find any eigenvectors.)
Problem 6. In each part, you are given a matrix $A$ and its eigenvalues. Find a basis for each of the eigenspaces of $A$ and determine if $A$ is diagonalizable. If so, find a diagonal matrix $D$ and an invertible matrix $P$ so that $P^{-1}AP = D$.

A. The matrix is

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -3 & -1 & 3 \\ -3 & 0 & 2 \end{bmatrix}$$

and the eigenvalues are $-1$ and $2$.

B. The matrix is

$$A = \begin{bmatrix} 11 & 10 & -22 \\ 9 & 11 & -21 \\ 9 & 9 & -19 \end{bmatrix}$$

and the eigenvalues are $-1$ and $2$.

Problem 7. In each part, you are given a matrix $A$ and its eigenvalues. Find a basis for each of the eigenspaces of $A$ and determine if $A$ is diagonalizable. If so, find a diagonal matrix $D$ and an invertible matrix $P$ so that $P^{-1}AP = D$. This problem will require the use of complex numbers.

A. The matrix is

$$A = \begin{bmatrix} -31 & 27 & -3 & -6 \\ -36 & 32 & -3 & -12 \\ 48 & -33 & 8 & -36 \\ -3 & 3 & 0 & -1 \end{bmatrix}$$

and the eigenvalues are $2 \pm 3i$.

B. The matrix is

$$A = \begin{bmatrix} -9 & 125 & -2 & -40 \\ -52 & 657 & -4 & -210 \\ 2 & 9 & 1 & -3 \\ -160 & 2018 & -12 & -645 \end{bmatrix}$$

and the eigenvalues are $1 \pm 2i$. 