• Write all of your answers on separate sheets of paper.

• This is a takehome exam. You may use the textbook and your class notes. You may discuss the problems with classmates, but write up your own solutions.

• You must show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).

• You may use Maple or a calculator to do integrals or algebra. Say where you have done so.

• This exam has 5 problems. There are 500 points total.

Good luck!
If you get something like $a_n = \frac{1 - (-1)^n}{n}$, don’t worry about dealing with the odd and even indices separately.

**Problem 1.** In this problem, find the solution of the following problem for the wave equation on the interval $0 \leq x \leq 2$:

$$\frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad t > 0,$$
$$u(0, t) = 0, \quad u(2, t) = 0, \quad t > 0,$$
$$u(x, 0) = x(2 - x), \quad 0 < x < 2,$$
$$u_t(x, 0) = x(1 - x)(2 - x), \quad 0 < x < 2.$$

**Problem 2.** In this problem, find the solution of the following problem for a modified heat equation on the interval $[0, \pi]$, where the ends of the rod are insulated.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u, \quad 0 < x < \pi,$$
$$u_x(0, t) = 0, \quad u_x(\pi, t) = 0, \quad t > 0,$$
$$u(x, 0) = x^2, \quad 0 < x < \pi.$$

**Problem 3.** Solve the following Dirichlet problem for the Laplace equation on the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 2, \quad 0 < y < 1,$$
$$u(x, 0) = 0, \quad u(x, 1) = x^2(2 - x), \quad 0 < x < 2,$$
$$u(0, y) = 0, \quad u(2, y) = 0, \quad 0 < y < 1.$$
Problem 4. Solve the following problem for an inhomogenous heat equation of the interval \([0, 1]\)

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6x \quad 0 < x < 1, \quad t > 0,
\]

\[
u(0, t) = 0, \quad u(1, t) = 2, \quad t > 0
\]

\[
u(x, 0) = x, \quad 0 < x < 1.
\]

Problem 5. Solve the following problem for the heat equation of the interval \([0, \pi]\)

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6x \quad 0 < x < \pi, \quad t > 0,
\]

\[(*)\]

\[
u(0, t) = 0, \quad u(\pi, t) = \pi t^2, \quad t > 0
\]

\[
u(x, 0) = 0, \quad 0 < x < \pi.
\]

Hint: Find a function \(\psi(x, t)\) so that \(\psi_{xx} = 0\) and \(\psi(x, t)\) satisfies the boundary conditions in \((*)\). Let \(u(x, t) = v(x, t) + \psi(x, t)\) and solve for \(v(x, t)\).