Exam 1

Work the Exam individually. Show your work in Maple. You may attach hand written work, if necessary.

> restart;
> with(LinearAlgebra):

**Problem 1**

100 points

The following matrix $N$ is nilpotent. Find $e^{Nt}$.

$$N := \begin{bmatrix} 7 & -28 & 63 & 35 & -70 \\ 16 & -64 & 164 & 70 & -135 \\ 3 & -12 & 31 & 13 & -25 \\ 14 & -45 & 57 & 76 & -150 \\ 4 & -11 & 0 & 25 & -50 \end{bmatrix}$$

**Problem 2**

400 points

In each part

1. Find the eigenvalues of the given matrix $A$.
2. Find the Jordan Decomposition $A = S + N$. 
3 Is the matrix diagonalizable?
4 Find $e^{St}$, $e^{Nt}$, and $e^{At}$
5 Solve the initial value problem $x' = Ax, x(0) = c$, for the given vector $c$
6 Consider the system of differential equations $x' = Ax$. Classify the fixed point at the origin as asymptotically stable, stable or unstable.

## Part A.

$$A := \text{Matrix}(4, 4, \{(1, 1) = 131, (1, 2) = -44, (1, 3) = 18, (1, 4) = 80, (2, 1) = 465, (2, 2) = -156, (2, 3) = 63, (2, 4) = 281, (3, 1) = -132, (3, 2) = 44, (3, 3) = -20, (3, 4) = -82, (4, 1) = 66, (4, 2) = -22, (4, 3) = 9, (4, 4) = 39\});$$

$$A := \begin{bmatrix} 131 & -44 & 18 & 80 \\ 465 & -156 & 63 & 281 \\ -132 & 44 & -20 & -82 \\ 66 & -22 & 9 & 39 \end{bmatrix} \quad (2)$$

$$c := <-1, 2, 1, -4>;$$

$$c := \begin{bmatrix} -1 \\ 2 \\ 1 \\ -4 \end{bmatrix} \quad (3)$$

## Part B.

$$A := \text{Matrix}(4, 4, \{(1, 1) = -33, (1, 2) = -2899, (1, 3) = 220, (1, 4) = 0, (2, 1) = 47, (2, 2) = 4373, (2, 3) = -332, (2, 4) = 0, (3, 1) = 615, (3, 2) = 57244, (3, 3) = -4346, (3, 4) = 0, (4, 1) = -216, (4, 2) = -20392, (4, 3) = 1548, (4, 4) = -1\});$$

$$A := \begin{bmatrix} -33 & -2899 & 220 & 0 \\ 47 & 4373 & -332 & 0 \\ 615 & 57244 & -4346 & 0 \\ -216 & -20392 & 1548 & -1 \end{bmatrix} \quad (4)$$

$$c := <-1, 2, 1, -4>;}$$
Part C.
\[
A := \text{Matrix}(4, 4, \{(1, 1) = 29, (1, 2) = -32, (1, 3) = -40, (1, 4) = -20, (2, 1) = 22, (2, 2) = -31, (2, 3) = -20, (2, 4) = -60, (3, 1) = 6, (3, 2) = -2, (3, 3) = -15, (3, 4) = 26, (4, 1) = -2, (4, 2) = 4, (4, 3) = 0, (4, 4) = 13\})
\]
\[
c := \langle 1, -1, 1, -1 \rangle;
\]

Part D.
\[
A := \text{Matrix}(4, 4, \{(1, 1) = 76, (1, 2) = -14, (1, 3) = -73, (1, 4) = -3, (2, 1) = 141, (2, 2) = -26, (2, 3) = -136, (2, 4) = -1, (3, 1) = 49, (3, 2) = -9, (3, 3) = -47, (3, 4) = -3, (4, 1) = -68, (4, 2) = 13, (4, 3) = 67, (4, 4) = 5\})
\]
\[
c := \langle 1, -1, 1, -1 \rangle;
**Problem 3**

200 points

In each part, find the fixed points of the given system of differential equations and classify each fixed point (if possible) according to the categories on page 398 of the book. Use the DEPlot command in Maple to show the direction field of the equations near the fixed points. You can use Maple to solve equations and do algebra.

**Part A.**

\[
\frac{dx}{dt} = x y - 3 y - 4 \\
\frac{dy}{dt} = y^2 - x^2
\]

**Part B.**

\[
\frac{dx}{dt} = x + xy - 3x^2 \\
\frac{dy}{dt} = 4y - 2xy - y^2
\]

**Problem 4**

100 points

Consider the system of differential equations \( x' = Ax \), where

\[
> A := <<r,-1>|<1,1>>;
\]
where $r$ is a real parameter. Discuss the stability of the origin as $r$ varies.

**Problem 5**

100 points

Consider inhomogenous system of differential equation $x'(t) = Ax(t) + f(t)$, where

$$A := \begin{bmatrix} r & 1 \\ -1 & 1 \end{bmatrix}$$

(10)

where $r$ is a real parameter. Discuss the stability of the origin as $r$ varies.

> $A := \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$

(11)

> $f := t \rightarrow \langle \exp(-t), \exp(-t) \rangle$

(12)

> $f(t)$;

$$\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$$

(13)

Use the variation of constants formula to find the solution of the system with the initial condition

> $c := \text{Vector}(2, \{(1) = 1, (2) = 2\})$;

(14)

> $c := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Here is $e^{At}$ as a maple function

> $\expA := t \rightarrow \text{Matrix}(2, 2, \{(1, 1) = \exp(-t)\cos(t), (1, 2) = -\exp(-t)\sin(t), (2, 1) = \exp(-t)\sin(t), (2, 2) = \exp(-t)\cos(t)\})$;

$\expA := t \rightarrow \text{Matrix}(2, 2, \{(1, 1) = e^{-t}\cos(t), (1, 2) = -e^{-t}\sin(t), (2, 1) = e^{-t}\sin(t), (2, 2) = e^{-t}\cos(t)\})$

(15)

> $\expA(t)$;

$$\begin{bmatrix} e^{-t}\cos(t) & -e^{-t}\sin(t) \\ e^{-t}\sin(t) & e^{-t}\cos(t) \end{bmatrix}$$

(16)

> $\expA(s)$;
\[
\begin{bmatrix}
e^{-s} \cos(s) & -e^{-s} \sin(s) \\
e^{-s} \sin(s) & e^{-s} \cos(s)
\end{bmatrix}
\]