EXAM

Practice Questions for the Final Exam

Math 3350, Spring 2004

May 3, 2004

ANSWERS
These are some practice problems from Chapter 10, Sections 1–4. See previous practice problem sets for the material before Chapter 10.

**Problem 1.** Let \( f(x) \) be the function of period \( 2L = 4 \) which is given on the interval \((-2, 2)\) by

\[
f(x) = \begin{cases} 
0, & -2 < x < 0 \\
2 - x, & 0 < x < 2.
\end{cases}
\]

Find the Fourier Series of \( f(x) \).

**Answer:**
The function is neither even nor odd. The Fourier coefficients are calculated as follows.

For \( a_0 \), we have

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx = \frac{1}{2} \int_{-2}^{2} f(x) \, dx = \frac{1}{4} \int_{0}^{2} (2 - x) \, dx = \frac{1}{2},
\]

since \( f(x)=0 \) on \((-2, 0)\).

For \( a_n \) with \( n \geq 1 \), we have

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left(\frac{n\pi}{L} x\right) \, dx = \frac{1}{2} \int_{-2}^{2} f(x) \cos \left(\frac{n\pi}{2} x\right) \, dx = \frac{1}{2} \int_{0}^{2} (2 - x) \cos \left(\frac{n\pi}{2} x\right) \, dx = -\frac{2}{n \pi^2} [1 - (-1)^n] = \begin{cases} 0, & n \text{ even} \\
-\frac{4}{n \pi^2}, & n \text{ odd.}
\end{cases}
\]
Finally, for $b_n$ we get

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) dx$$

$$= \frac{1}{2} \int_{-2}^{2} f(x) \sin \left( \frac{n\pi x}{2} \right) dx$$

$$= \frac{1}{2} \int_{0}^{2} (2-x) \sin \left( \frac{n\pi x}{2} \right) dx$$

$$= \frac{2}{n\pi}.$$

For the cosine terms in the series, we use $2k+1$ to run over the odd integers. Thus, the Fourier Series of $f(x)$ is

$$\frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos \left( \frac{(2k+1)\pi}{2} x \right) + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{n} \sin \left( \frac{k\pi}{2} x \right).$$

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**Problem 2.** Let $f(x)$ be the function of period $2L = 2$ which is given on the interval $(-1,1)$ by $f(x) = 1 - x^2$.

Find the Fourier Series of $f(x)$.

**Answer:**

The function is even.

Using the fact that the function is even, we get

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 - x^2) dx$$

$$= \frac{1}{2} \int_{0}^{1} (1 - x^2) dx$$

$$= \frac{2}{3}.$$

Again using the fact that the function is even, we get

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) dx$$

$$= \int_{-1}^{1} (1 - x^2) \cos(n\pi x) dx$$

$$= 2 \int_{0}^{1} (1 - x^2) \cos(n\pi x) dx$$

$$= \frac{4(-1)^{n+1}}{n^2\pi^2}.$$
For the \(b_n\)'s, we have

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx
\]

\[
= \int_{-1}^{1} (1 - x^2) \sin(n\pi x) dx
\]

\[
= 0,
\]

because the integrand is an odd function.

Thus, the Fourier Series of \(f(x)\) is

\[
\frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(n\pi x).
\]

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**Problem 3.** Consider the function

\[
f(x) = 2x, \quad 0 < x < 1.
\]

A. Find the Fourier cosine series of \(f(x)\) Hint: you’re using the even half-range expansion.

*Answer:*

In this case \((0, L) = (0, 1)\), so \(L = 1\). Using the formulas for the even half-range expansion, we get the following.

For \(a_0\),

\[
a_0 = \frac{1}{L} \int_{0}^{L} f(x) \, dx
\]

\[
= \int_{0}^{2} 2x \, dx
\]

\[
= 1.
\]

For \(a_n\) we get

\[
a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx
\]

\[
= 2 \int_{0}^{2} 2x \cos(n\pi x) \, dx
\]

\[
= \frac{-4}{n^2\pi^2} \left[ 1 - (-1)^n \right]
\]

\[
= \begin{cases} 
0, & n \text{ even} \\
-\frac{8}{n^2\pi^2}, & n \text{ odd}.
\end{cases}
\]
Using $2k + 1, k = 0, 1, 2, \ldots$ to range over the odd integers, the Fourier cosine series of $f(x)$ is

$$1 - \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos((2k + 1)\pi x).$$

B. Find the Fourier sine series of $f(x)$. Hint: you’re using the odd half-range expansion.

*Answer:*

Using the formulas for the odd half-range expansion, we have

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi}{L} x \right) dx$$

$$= 2 \int_0^1 2x \sin(n\pi x) dx$$

$$= \frac{4(-1)^{n+1}}{n\pi}.$$ 

so the Fourier sine series of $f(x)$ is

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x).$$