Problem 1. Find the general solution by the method of undetermined coefficients:

\[(D^2 + 4)y = x \sin(2x).\]

**Answer:**
Recall that

\[e^{(\alpha + i\beta)x} = e^{\alpha x} \cos(\beta x) + ie^{\alpha x} \sin(\beta x).\]

Using this formula, we see that \(\sin(2x)\) is the imaginary part of \(e^{2ix}\). Thus, \(x \sin(2x)\) is the imaginary part of \(xe^{2ix}\). All we need to do is solve the complex equation

\[(\bigtriangledown)\quad (D^2 + 4)y = xe^{2ix}\]

and then take the imaginary part of our solution.

To solve \((\bigtriangledown)\) move the exponential to the left side and write the equation as

\[e^{-2ix}P(D)y = x,\]

where \(P(D) = D^2 + 4\). Applying the Shifting Rule, we get

\[P(D + 2i)[e^{-2ix}y] = x.\]

We set \(z = e^{-2ix}y\) and calculate that

\[P(D + 2i) = (D + 2i)^2 + 4\]
\[= D^2 + 4iD - 4 + 4\]
\[= D^2 + 4iD.\]

Thus, the equation for \(z\) is

\[(D^2 + 4iD)z = x.\]

Since we can factor out a \(D\), we do so and rewrite the equation as

\[(D + 4i)[Dz] = x.\]

Let \(w = Dz\), so the equation for \(w\) is

\[(D + 4i)w = x.\]

This equation has a nonzero constant term in the operator and a polynomial on the right-hand side, so we can solve it by undetermined coefficients. Since the right side is a polynomial of degree 1, we should choose a polynomial of degree 1 for our trial solution, say \(w = Ax + B\). Substituting this in the differential equation yields the equation

\[A + 4i[Ax + B] = x.\]
Collecting coefficients in this equation gives

\[ 4iAx + (A + 4iB) = x. \]

Equating coefficients of powers of \( x \) in this equation, we get the equations

\[ 4iA = 1, \quad A + 4iB = 0. \]

The solution of these equations is

\[ A = -\frac{1}{4}i, \quad B = \frac{1}{16}. \]

Thus, we have

\[ w = -\frac{1}{4}ix + \frac{1}{16}. \]

Since \( w = Dz \), we can find \( z \) by integrating \( w \). Thus,

\[ z = -\frac{1}{8}ix^2 + \frac{1}{16}x. \]

Since \( z = e^{-2ix}y \), we have

\[ y = \left( -\frac{1}{8}ix^2 + \frac{1}{16}x \right) e^{2ix} \]

as a particular solution of equation (\( \nabla \)). To find the solution of our original equation, we need to work out the real and imaginary parts of \( y \). We have

\[
\begin{align*}
y &= \left( -\frac{1}{8}ix^2 + \frac{1}{16}x \right) e^{2ix} \\
&= \left( -\frac{1}{8}ix^2 + \frac{1}{16}x \right) (\cos(2x) + i\sin(2x)) \\
&= \frac{1}{16} \cos(2x) + \frac{1}{8}x^2 \sin(2x) + i \left[ -\frac{1}{8}x^2 \cos(2x) + \frac{1}{16}x \sin(2x) \right].
\end{align*}
\]

To get a solution of our original equation, we take the imaginary part of this. Thus, we have

\[ y_p = -\frac{1}{8}x^2 \cos(2x) + \frac{1}{16}x \sin(2x) \]

as a particular solution of our original equation.

The solution of the homogeneous equation \((D^2+4)y = 0\) is \( y_h = C_1 \cos(2x) + C_2 \sin(2x) \), so the general solution of the equation given in the problem is

\[ y = -\frac{1}{8}x^2 \cos(2x) + \frac{1}{16}x \sin(2x) + C_1 \cos(2x) + C_2 \sin(2x). \]
Problem 2. Find the Laplace Transform of the following function:

\[
\begin{cases}
  0, & 0 < t < 1 \\
  t^2 + 2t, & 1 < t < 2 \\
  1, & 2 < t < \infty
\end{cases}
\]

Answer:
Using the indicator functions of the intervals, we can write \( f(t) \) as

\[
\begin{align*}
  f(t) &= 0 \cdot I_{(0,1)}(t) + (t^2 + 2t) \cdot I_{(1,2)}(t) + 1 \cdot I_{(2,\infty)}(t) \\
        &= (t^2 + 2t) \cdot I_{(1,2)}(t) + I_{(2,\infty)}(t)
\end{align*}
\]

From our table of indicator functions, we have

\[
\begin{align*}
  I_{(1,2)}(t) &= u(t - 1) - u(t - 2) \\
  I_{(2,\infty)}(t) &= u(t - 2)
\end{align*}
\]

Plugging this in, we have

\[
f(t) = (t^2 + 2t)[u(t - 1) - u(t - 2)] + u(t - 2).
\]

Collecting coefficients of the \( u \)'s, this is

\[
(2.1) \quad f(t) = u(t - 1)[t^2 + 2t] + u(t - 2)[1 - 2t - t^2].
\]

To find the transform of this, we want to use the shifting rule

\[
(2.2) \quad \mathcal{L}[u(t - a)g(t - a)] = e^{-as}G(s), \quad G(s) = \mathcal{L}[g(t)].
\]

Consider the first term in (2.1), \( u(t - 1)[t^2 + 2t] \). To get this to match the left-hand side of (2.2), we must have \( u(t - a) = u(t - 1) \), so \( a = 1 \), and \( g(t - 1) = t^2 + 2t \). From this, we have to figure out what \( g(t) \) is. To do this, we substitute \( t + 1 \) for \( t \), and get

\[
\begin{align*}
  g(t) &= g(t + 1 - 1) \\
        &= (t + 1)^2 + 2(t + 1) \\
        &= t^2 + 2t + 1 + 2t + 2 \\
        &= t^2 + 4t + 3.
\end{align*}
\]

Then we have

\[
G(s) = \mathcal{L}[t^2 + 4t + 3] = \frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}.
\]

Applying (2.2) we then get

\[
\mathcal{L}[u(t - 1)[t^2 + 2t]] = e^{-s}G(s) = e^{-s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right].
\]

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Similarly, consider the second term in (2.1), \(u(t - 2)[1 - 2t - t^2]\). Comparing this with (2.2) we have \(a = 2\) and \(g(t - 2) = 1 - 2t - t^2\). Thus,

\[
g(t) = g(t + 2 - 2) = 1 - 2(t + 2) - (t + 2)^2 = 1 - 2t - 4 - t^2 - 4t - 4 = -7 - 8t - t^2.
\]

Then

\[
G(s) = \mathcal{L}[-7 - 8t - t^2] = -\frac{7}{s} - \frac{8}{s^2} - \frac{2}{s^3}.
\]

and using (2.2) we get

\[
\mathcal{L}[u(t - 2)[1 - 2t - t^2]] = -e^{-2s}\left[\frac{7}{s} + \frac{8}{s^2} + \frac{2}{s^3}\right].
\]

Combining these results, we get

\[
\mathcal{L}[f(t)] = e^{-s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right] - e^{-2s}\left[\frac{7}{s} + \frac{8}{s^2} + \frac{2}{s^3}\right]
\]

**Problem 3.** Find the Inverse Laplace Transform of the following function:

\[
F(s) = \frac{1}{s + 1} + e^{-s}\frac{1}{(s + 2)^2} + e^{-3s}\frac{1}{s^2 + 4}
\]

**Answer:**

We know

\begin{align*}
(3.1) & \quad \mathcal{L}^{-1}\left[\frac{1}{s + 1}\right] = e^{-t}. \\
(3.2) & \quad \mathcal{L}^{-1}\left[\frac{1}{(s + 2)^2}\right] = te^{-2t}. \\
(3.3) & \quad \mathcal{L}^{-1}\left[\frac{1}{s^2 + 4}\right] = \frac{1}{2} \mathcal{L}^{-1}\left[\frac{2}{s^2 + 4}\right] = \frac{1}{2} \sin(2t).
\end{align*}

We use the shifting rule:

\[
\mathcal{L}^{-1}[e^{-as}F(s)] = u(t - a)f(t - a), \quad f(t) = \mathcal{L}^{-1}[F(s)].
\]

Applying this rule, and (3.2) we get

\[
\mathcal{L}^{-1}\left[e^{-s}\frac{1}{(s + 2)^2}\right] = u(t - 1)(t - 1)e^{-2(t - 1)}
\]
and from (3.3) we get
\[ \mathcal{L}^{-1} \left[ e^{-3s} \frac{1}{s^2 + 4} \right] = \frac{1}{2} u(t - 3) \sin(2(t - 3)). \]

Thus, we get
\[ \mathcal{L}^{-1}[F(s)] = e^{-t} + u(t - 1)(t - 1)e^{-2(t-1)} + \frac{1}{2} u(t - 3) \sin(2(t - 3)). \]

---

**Problem 4.** Solve the following initial value problem, using Laplace Transforms:

(4.1) \[ y'' - 2y' + y = t^2 u(t - 1) + \delta(t - 2), \quad y(0) = 0, \quad y'(0) = 1. \]

**Answer:**

Let's start by finding the Laplace transform of the right-hand side. We know that
\[ (4.2) \quad \mathcal{L}[\delta(t - 2)] = e^{-2s}. \]

To find the transform of the other term, recall the shifting rule
\[ (4.3) \quad \mathcal{L}[u(t - a)f(t - a)] = e^{-as}F(s), \quad F(s) = \mathcal{L}[f(t)]. \]

To match the left-hand side of this with \( u(t - 1)t^2 \) we must have \( a = 1 \) and \( f(t-1) = t^2 \). To find \( f(t) \), substitute \( t+1 \) for \( t \) in the equation \( f(t-1) = t^2 \). This gives \( f(t) = (t+1)^2 = t^2+2t+1 \). Thus, \( F(s) = \mathcal{L}[t^2+2t+1] = 2/s^3+2/s^2+1/s \).

Hence, by the shifting rule, we have
\[ (4.4) \quad \mathcal{L}[u(t - 1)t^2] = e^{-s} \left[ \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right] = e^{-s} \frac{s^2 + 2s + 1}{s^3}. \]

Now transform both sides of (4.1). The result is
\[ s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + Y(s) = e^{-s} \frac{s^2 + 2s + 1}{s^3} + e^{-2s}, \]

by (4.2) and (4.4). Putting in the initial conditions and collecting terms gives us
\[ (s^2 - 2s + 1)Y(s) - 1 = e^{-s} \frac{s^2 + 2s + 1}{s^3} + e^{-2s}. \]

Solving this for \( Y(s) \), we get
\[ Y(s) = \frac{1}{s^2 - 2s + 1} + e^{-s} \frac{s^2 + 2s + 1}{s^3(s^2 - 2s + 1)} + e^{-2s} \frac{1}{s^2 - 2s + 1} \]
\[ = \frac{1}{(s - 1)^2} + e^{-s} \frac{s^2 + 2s + 1}{s^3(s - 1)^2} + e^{-2s} \frac{1}{(s - 1)^2}. \]
We know, of course,
\[ \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] = te^t. \]

To deal with the middle term, we find (by machine) the partial fractions decomposition
\[ \frac{s^2 + 2s + 2}{s^3(s-1)^2} = -11\frac{1}{s-1} + 5\frac{1}{(s-1)^2} + 11\frac{1}{s} + \frac{2}{s^3} + 6\frac{1}{s^2}. \]

Thus, the inverse transform is
\[ f(t) = \mathcal{L}^{-1}\left[\frac{s^2 + 2s + 2}{s^3(s-1)^2}\right] = -11e^t + 5te^t + 11 + 6t + t^2. \]

By the shifting rule (4.3), we then get
\[ \mathcal{L}^{-1}\left[e^{-s}\frac{s^2 + 2s + 2}{s^3(s-1)^2}\right] = u(t-1)f(t-1) \]
\[ = u(t-1)[(t-1)^2 + 6(t-1) + 11 + 5(t-1)e^{t-1} - 11e^{t-1}]. \]

Similarly,
\[ \mathcal{L}^{-1}\left[e^{-2s}\frac{1}{(s-1)^2}\right] = u(t-2)(t-2)e^{-2}. \]

Thus, finally,
\[ y(t) = te^t + u(t-1)[(t-1)^2 + 6(t-1) + 11 + 5(t-1)e^{t-1} - 11e^{t-1}] + u(t-2)(t-2)e^{-2}. \]

---

**Problem 5.** In the following problems, use formulas (1) and (6) from section 5.4.

1. Find the Laplace transform of \( f(t) = te^{2t} \cos(t) \).
2. Find the Laplace transform of \( f(t) = (e^{2t} - 1)/t \).
3. Find the inverse Laplace transform of the function
\[ G(s) = \ln\left(\frac{s^2 + 1}{s^2 + 4}\right). \]

**Answer:**
The formulas referred to are
\[ \mathcal{L}[tf(t)] = -F'(s), \quad F(s) = \mathcal{L}[f(t)], \]
and

$$L\left[ \frac{f(t)}{t} \right] = \int_s^\infty F(\sigma) \, d\sigma, \quad F(s) = L[f(t)],$$

provided that \( \lim_{t \to 0^+} f(t)/t \) exists.

For the first part, we apply (5.1) with \( f(t) = e^{2t} \cos(t) \). From the table, we have

$$L[e^{2t} \cos(t)] = \frac{s - 2}{(s - 2)^2 + 1}.$$  

Thus, from (5.1),

$$L[te^{2t} \cos(t)] = -\frac{d}{ds} \frac{s - 2}{(s - 2)^2 + 1} = \frac{s^2 - 4s + 3}{(s - 2)^2 + 1}.$$  

For the second part, note that \( \lim_{t \to 0^+} (e^{2t} - 1)/t = 2 \) (use L’Hôpital’s Rule), so we can apply (5.2) with \( f(t) = e^{2t} - 1 \). Then

$$F(s) = L[e^{2t} - 1] = \frac{1}{s - 2} - \frac{1}{s}.$$  

Thus, by (5.2) we have

$$L\left[ \frac{e^{2t} - 1}{t} \right] = \int_s^\infty F(\sigma) \, d\sigma$$

$$= \int_s^\infty \left[ \frac{1}{\sigma - 2} - \frac{1}{\sigma} \right] \, d\sigma$$

$$= \left[ \ln(\sigma - 2) - \ln(\sigma) \right]_{\sigma = \infty}^{\sigma = s}$$

(loosely speaking)

$$= \ln\left( \frac{\sigma - 2}{\sigma} \right)_{\sigma = 0}^{\sigma = \infty}$$

$$= \ln(1 - 2/\sigma)_{\sigma = s}^{\sigma = \infty}$$

$$= 0 - \ln(1 - 2/s)$$

$$= - \ln(1 - 2/s)$$

since

$$\lim_{\sigma \to \infty} \ln(1 - 2/\sigma) = \ln(1) = 0.$$  

For the third part, set

$$G(s) = \ln\left[ \frac{s^2 + 1}{s^2 + 4} \right].$$  

Then

$$G(s) = \ln(s^2 + 1) - \ln(s^2 + 4),$$  

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so we have
\[
F(s) = G'(s) = \frac{2s}{s^2 + 1} - \frac{2s}{s^2 + 4}.
\]
(this equation defines \(F(s)\)).

From this we have
\[
\int_s^\infty F(\sigma) d\sigma = \int_s^\infty G'(\sigma) d\sigma = G(\infty) - G(s) = -G(s),
\]
(loosely speaking) since \(G(\sigma) \to 0\) as \(\sigma \to \infty\). Thus, we have
\[
G(s) = -\int_s^\infty F(\sigma)d\sigma.
\]
From the inverse version of (5.2), we then have
\[
g(t) = \mathcal{L}^{-1}[G(s)] = -\mathcal{L}^{-1} \left[ \int_s^\infty F(\sigma)d\sigma \right] = -\frac{f(t)}{t},
\]
where \(f(t) = \mathcal{L}^{-1}[F(s)]\). From the table and (5.3),
\[
f(t) = 2\sin(t) - 2\sin(2t).
\]
Thus, finally, the answer is
\[
g(t) = 2\frac{\sin(t) - \sin(2t)}{t}.
\]
You can verify that \(\lim_{t \to 0^+} g(t)\) exists by L'Hôpital's Rule.

**Problem 6.** Find the convolution \(t^2 \ast t\) directly from the definition.

**Answer:**
The definition of convolution is
\[
(f \ast g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau.
\]
Let \(f(t) = t^2\) and \(g(t) = t\). Plugging into the formula above, we have
\[
(f \ast g)(t) = \int_0^t \tau^2(t - \tau) d\tau
\]
\[
= \int_0^t (t\tau^2 - \tau^3) d\tau
\]
\[
= \left[ \frac{1}{3}t\tau^3 - \frac{1}{4}\tau^4 \right]_{\tau=0}^{\tau=t}
\]
\[
= \frac{1}{3}t^4 - \frac{1}{4}t^4 - [0 - 0]
\]
\[
= \frac{1}{12}t^4.
\]
Problem 7. Find the convolution \( \cos(2t) \ast \sin(t) \) using Laplace transforms.

Answer:
We want to use the formula
\[
(7.1) \quad \mathcal{L}[(f \ast g)(t)] = F(s)G(s).
\]
We know, of course, that
\[
\mathcal{L}[\cos(2t)] = \frac{s}{s^2 + 4} \quad \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1}.
\]
Thus, by (7.1),
\[
\mathcal{L}[\cos(2t) \ast \sin(t)] = \frac{s}{s^2 + 4} \frac{1}{s^2 + 1} = \frac{s}{(s^2 + 4)(s^2 + 1)}.
\]
All we have to do is take the inverse transform of the right-hand side. The partial fractions decomposition is (by machine)
\[
\frac{s}{(s^2 + 4)(s^2 + 1)} = \frac{1}{3 s^2 + 1} - \frac{1}{3 s^2 + 4}.
\]
Thus, we have
\[
\cos(2t) \ast \sin(t) = \mathcal{L}^{-1}\left[ \frac{s}{(s^2 + 4)(s^2 + 1)} \right] \\
= \frac{1}{3} \mathcal{L}^{-1}\left[ \frac{s}{s^2 + 1} \right] - \frac{1}{3} \mathcal{L}^{-1}\left[ \frac{s}{s^2 + 4} \right] \\
= \frac{1}{3} \cos(t) - \frac{1}{3} \cos(2t).
\]

Problem 8. Express the solution of the following initial value problem using a convolution integral.

\[ y'' - 2y' + y = r(t), \quad y(0) = 0, \quad y'(0) = 1. \]

Answer:
Take the Laplace Transform of both sides of the equation. This gives
\[
s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + Y(s) = R(s),
\]
where \( R(s) = \mathcal{L}[r(t)] \). Plugging in the initial conditions and simplifying we get
\[
(s^2 - 2s + 1)Y(s) = 1 + R(s).
\]
Solving this equation for $Y(s)$ gives us

$$Y(s) = \frac{1}{s^2 - 2s + 1} + \frac{R(s)}{s^2 - 2s + 1} = \frac{1}{(s - 1)^2} + \frac{R(s)}{(s - 1)^2}.$$ 

As we know,

$$\mathcal{L}^{-1}\left[\frac{1}{(s - 1)^2}\right] = te^t.$$

To take the inverse transform of the other term, we use the formula

$$\mathcal{L}^{-1}[F(s)G(s)] = (f * g)(t).$$

Thus,

$$\mathcal{L}^{-1}\left[R(s)\frac{1}{(s - 1)^2}\right] = r(t) * te^t.$$

From the definition of convolution,

$$r(t) * te^t = \int_0^t r(\tau)(t - \tau)e^{t-\tau} d\tau.$$ 

Putting all this together, the solution of our initial value problem is

$$y(t) = \int_0^t (t - \tau)e^{t-\tau}r(\tau) d\tau + te^t.$$