Problem Set

Problem Set #1
Math 5322, Fall 2001
March 4, 2002

ANSWERS
All of the problems are from Chapter 3 of the text.

Problem 1. [Problem 2, page 88]

If \( \nu \) is a signed measure, \( E \) is \( \nu \)-null iff \( |\nu|(E) = 0 \). Also, if \( \nu \) and \( \mu \) are signed measures, \( \nu \perp \mu \) iff \( |\nu| \perp \mu \) and \( \nu^- \perp \mu \)

**Answer:**

For the first part of the problem, let \( X = P \cup N \) be a Hahn decomposition of \( X \) with respect to \( \nu \). Thus, we have

\[
\nu^+(E) = \nu(E \cap P) \\
\nu^-(E) = -\nu(E \cap N) \\
|\nu| = \nu^+ + \nu^-
\]

Assume that \( E \) is \( \nu \)-null. This means that for all measurable \( F \subseteq E, \nu(F) = 0 \). But then \( E \cap P \subseteq E \), so \( \nu^+(E) = 0 \) and \( E \cap N \subseteq E \), so \( \nu^-(E) = 0 \). Then we have \( |\nu|(E) = \nu^+(E) + \nu^-(E) = 0 \)

Conversely, suppose that \( |\nu|(E) = 0 \). Let \( F \subseteq E \) be measurable. Then \( 0 \leq \nu^+(F) + \nu^-(F) = |\nu|(F) \leq |\nu|(E) = 0 \), so \( \nu^+(F) = 0 \) and \( \nu^-(F) = 0 \). But then \( \nu(F) = \nu^+(F) - \nu^-(F) = 0 \). Thus, we can conclude that \( E \) is \( \nu \)-null.

For the second part of the problem, we want to show that the following conditions are equivalent.

1. \( \nu \perp \mu \)
2. \( \nu^+ \perp \mu \) and \( \nu^- \perp \mu \)
3. \( |\nu| \perp \mu \)

Let’s first show that (1) \( \implies \) (2). Since \( \nu \perp \mu \), we can decompose \( X \) into a disjoint union of measurable sets \( A \) and \( B \) so that \( A \) is \( \mu \)-null and \( B \) is \( \nu \)-null.

We can also find an Hahn decomposition \( X = P \cup N \) with respect to \( \nu \), as above. Since \( B \) is \( \nu \)-null, we have \( \nu^+(B) = \nu(P \cap B) = 0 \) and \( \nu^-(B) = -\nu(B \cap N) = 0 \). Thus, \( B \) is \( \nu^+ \)-null and \( \nu^- \)-null. Since \( A \) is still \( \mu \)-null, we conclude that \( \nu^+ \perp \mu \) and \( \nu^- \perp \mu \).

Next, we show that (2) \( \implies \) (3). Since \( \nu^+ \perp \mu \), we can find disjoint measurable sets \( A_1 \) and \( B_1 \) so that \( X = A_1 \cup B_1, \nu^+(B_1) = 0 \) and \( A_1 \) is \( \mu \)-null. Similarly, we can write \( X = A_2 \cup B_2 \) (disjoint union) where \( \nu^-(B_2) = 0 \) and \( A_2 \) is \( \mu \)-null. We can then partition \( X \) as

\[
X = (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (A_2 \cap B_1) \cup (B_1 \cap B_2)
\]

Now, \( (A_1 \cap A_2) \) is contained in the \( \mu \)-null set \( A_1 \), and so is \( \mu \)-null. Similarly, \( A_2 \cap B_2 \) and \( A_2 \cap B_1 \) are \( \mu \)-null. We have \( 0 \leq \nu^+(B_1 \cap B_2) \leq \nu^+(B_1) = 0 \), so \( \nu^+(B_1 \cap B_2) = 0 \). Similarly, \( \nu^-(B_1 \cap B_2) = 0 \), and thus \( |\nu|(B_1 \cap B_2) = 0 \). We thus have a disjoint measurable decomposition \( X = A_3 \cup B_3 \) where

\[
A_3 = (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (A_2 \cap B_1) \\
B_3 = B_1 \cap B_2.
\]
The set $A_3$ is $\mu$-null and $B_3$ is $|\nu|$-null. This shows that $|\nu| \perp \mu$.

Finally we show that $(3) \implies (1)$. So, suppose that $|\nu| \perp \mu$. The we have a measurable decomposition $X = A \cup B$ where $|\nu|(B) = 0$ and $A$ is $\mu$-null. However, $|\nu|(B) = 0$ implies that $B$ is $\nu$-null (as we proved above), so we have $\nu \perp \mu$.

Problem 2. [Problem 3, page 88]

Let $\nu$ be a signed measure on $(X, \mathcal{M})$.

a. $L^1(\nu) = L^1(|\nu|)$.

b. If $f \in L^1(\nu)$,

\[
\left| \int f \, d\nu \right| \leq \int |f| \, d|\nu|.
\]

c. If $E \in \mathcal{M}$,

\[
|\nu|(E) = \sup \left\{ \left| \int_E f \, d\nu \right| \mid |f| \leq 1 \right\}
\]

Answer:
According to the definition on page 88 of the text, the definition of $L^1(\nu)$ is $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$ and if $f \in L^1(\nu)$, we define

\[
\int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-.
\]

So, consider part (a.) of the problem. If $f \in L^1(\nu)$, then, by definition the integrals

\[
(*) \quad \int |f| \, d\nu^+, \quad \int |f| \, d\nu^-
\]

are finite. On the other hand, $|\nu|$ is defined by $|\nu|(E) = \nu^+(E) + \nu^-(E)$. Hence, the equation

\[
\int \chi_E \, d|\nu| = \int \chi_E \, d\nu^+ + \int \chi_E \, d\nu^-
\]

holds by definition. Thus, we can apply the usual argument to show that

\[
(**) \quad \int g \, d|\nu| = \int g \, d\nu^+ + \int g \, d\nu^-
\]

holds for all measurable $g: X \to [0, \infty]$: This equations holds for characteristic functions, hence (by linearity) for simple functions and hence (by the monotone convergence theorem) for nonnegative functions. Thus, if the integrals in $(*)$ are finite, the integral $\int |f| \, d|\nu|$ is finite, so $f \in L^1(|\nu|)$.

Conversely, if $f \in L^1(|\nu|)$, the integral $\int |f| \, d|\nu|$ is finite, and so by $(**)$, the integrals in $(*)$ are finite, so $f \in L^1(\nu)$. 

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Next consider part (b.) of the problem. If $f \in L^1(\nu)$, then by definition

$$
\int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^- = \int f^+ \, d\nu^+ - \int f^- \, d\nu^- + \int f^+ \, d\nu^- + \int f^- \, d\nu^-,
$$

where all of the integrals on the right are positive numbers. Hence, by the triangle inequality,

$$
\left| \int f \, d\nu \right| \leq \int f^+ \, d\nu^+ + \int f^- \, d\nu^- + \int f^+ \, d\nu^- + \int f^- \, d\nu^- = \int |f| \, d\nu^+ + \int |f| \, d\nu^- = \int |f| \, d|\nu|,
$$

Finally, consider part (c.) of the problem. If $|f| \leq 1$, then for any $E \in \mathcal{M}$, $|f|\chi_E \leq \chi_E$. Thus, we have

$$
\left| \int_E f \, d\nu \right| \leq \int_E |f| \, d|\nu| = \int_X |f| \chi_E \, d|\nu| \leq \int_X \chi_E \, d|\nu| = |\nu|(E).
$$

Thus, the sup in part (c.) is $\leq |\nu|(E)$.

To get the reverse inequality, let $X = P \cup N$ be a Hahn decomposition of $X$ with respect to $\nu$, and recall the description of $\nu^+$ and $\nu^-$ in terms of $P$ and $N$ from the last problem. Then we have

$$
|\nu|(E) = \nu(E \cap P) - \nu(E \cap N) = \int (\chi_{E \cap P} - \chi_{E \cap N}) \, d\nu.
$$

But, $|\chi_{E \cap P} - \chi_{E \cap N}| \leq 1$ ($P$ and $N$ are disjoint). This shows that $|\nu|(E)$ is an element of the set we’re supping over, so $|\nu|(E) \leq$ the sup.

**Problem 3.** [Problem 4, page 88]

If $\nu$ is a signed measure and $\lambda, \mu$ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$. 

**Answer:**

Let $X = P \cup N$ be a Hahn decomposition of $X$ with respect to $\nu$. Then for any measurable set $E$, we have

$$
(\star) \quad \nu^+(E) = \nu(E \cap P) = \lambda(E \cap P) - \mu(E \cap P).
$$
If $\nu^+(E) = \infty$, this equation shows $\lambda(E \cap P) = \infty$, and so $\lambda(E) = \infty$. Thus, $\nu^+(E) \geq \lambda(E)$ is true in this case. If $\nu^+(E) < \infty$, then both numbers on the right of $(\ast)$ must be finite, and $(\ast)$ gives $\nu^+(E) + \mu(E \cap P) = \lambda(E \cap P)$. Thus, we have

$$\nu^+(E) \leq \nu^+(E) + \mu(E \cap P) = \lambda(E \cap P) \leq \lambda(E).$$

The argument for the other inequality is similar. We have

$$\nu^-(E) = -\nu(E \cap N) = -\lambda(E \cap N) + \mu(E \cap N)$$

If $\nu^-(E) = \infty$, then we must have $\mu(E) \geq \mu(E \cap N) = \infty$, so $\mu(E) \geq \nu^-(E)$.

Otherwise, every thing must be finite and

$$\nu^-(E) \leq \nu^-(E) + \lambda(E \cap N) = \mu(E \cap N) \leq \mu(E).$$

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**Problem 4.** [Problem 5, page 88] If $\nu_1$ and $\nu_2$ are signed measures that both omitted the value $+\infty$ or $-\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Use Exercise 4.)

**Answer:**

The hypothesis that $\pm \infty$ is omitted assures that the signed measure $\nu_1 + \nu_2$ is defined.

We have $\nu_1 = \nu_1^+ - \nu_1^-$ and $\nu_2 = \nu_2^+ - \nu_2^-$. Adding these equations gives

$$\nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-),$$

where the expressions in parentheses are positive measures. Thus, by Exercise 4, we have

$$\begin{align*}
(\nu_1 + \nu_2)^+ &\leq \nu_1^+ + \nu_2^+ \\
(\nu_1 + \nu_2)^- &\leq \nu_1^- + \nu_2^-
\end{align*}$$

and adding these inequalities gives $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

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**Problem 5.** [Problem 12, page 92] For $j = 1, 2$, let $\mu_j, \nu_j$ be $\sigma$-finite measures on $(X_j, M_j)$ such that $\nu_j \ll \mu_j$, then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2).$$

**Answer:**

We first want to show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$. So, suppose that $E \in M_1 \otimes M_2$ and $(\mu_1 \times \mu_2)(E) = 0$. Recall that the $x_1$-section of $E$ is defined by

$$E_{x_1} = \{ x_2 \in X_2 \mid (x_1, x_2) \in E \}.$$
By Theorem 2.36 on page 66 of the text, the function $x_1 \mapsto \mu_2(E_{x_1})$ is measurable and
\[ 0 = (\mu_1 \times \mu_2)(E) = \int_{X_1} \mu_2(E_{x_1}) \, d\mu_1(x_1) \]
Thus, $\mu_2(E_{x_1}) = 0$ for $\mu_1$-almost all $x_1$. In other words, there is a $\mu_1$-null set $N \subseteq X_1$ so that $\mu_2(E_{x_1}) = 0$ if $x_1 \notin N$. If $x_1 \notin N$, we must have $\nu_2(E_{x_1}) = 0$, since $\nu_2 \ll \mu_2$. We also have $\nu_1(N) = 0$, since $\nu_1 \ll \mu_1$. Thus, $x_1 \mapsto \nu_2(E_{x_1})$ is zero almost everywhere with respect to $\nu_1$. Thus, we have
\[ (\nu_1 \times \nu_2)(E) = \int_{X_1} \nu_2(E_{x_1}) \, d\nu_1(x_1) = 0. \]
Thus, $(\mu_1 \times \mu_2)(E) = 0 \implies (\nu_1 \times \nu_2)(E) = 0$, so $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$.

For notational simplicity, let
\[ f = \frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}, \quad f_1 = \frac{d\nu_1}{d\mu_1}, \quad f_2 = \frac{d\nu_2}{d\mu_2}. \]
The, we have
\[ (\nu_1 \times \nu_2)(E) = \int_E f(x_1, x_2) \, d(\mu_1 \times \mu_2)(x_1, x_2) \]
and $f$ is the unique (up to equality almost everywhere) function with this property. Of course we have
\[ \nu_1(A) = \int_A f_1(x_1) \, d\mu_1(x_1) \]
\[ \nu_2(B) = \int_B f_2(x_2) \, d\mu_2(x_2) \]
if $A \subseteq X_1$ and $B \subseteq X_2$ are measurable.

On the other hand, the functions $(x_1, x_2) \mapsto f_1(x_1)$ and $(x_1, x_2) \mapsto f_2(x_2)$ are measurable, so we can define a measure $\lambda$ on $X_1 \times X_2$ by
\[ \lambda(E) = \int_E f_1(x_1) f_2(x_2) \, d(\mu_1 \times \mu_2)(x_1, x_2). \]
Suppose that $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$. Then, we can use Tonelli’s Theorem to
calculate \( \lambda(A_1 \times A_2) \) as follows:

\[
\lambda(A_1 \times A_2) = \int_{A_1 \times A_2} f_1(x_1)f_2(x_2) \, d(\mu_1 \times \mu_2)(x_1, x_2)
\]

\[
= \int_{X_1 \times X_2} \chi_{A_1 \times A_2}(x_1, x_2) f_1(x_1)f_2(x_2) \, d(\mu_1 \times \mu_2)(x_1, x_2)
\]

\[
= \int_{X_1 \times X_2} \chi_{A_1}(x_1)\chi_{A_2}(x_2) f_1(x_1)f_2(x_2) \, d(\mu_1 \times \mu_2)(x_1, x_2)
\]

\[
= \int_{X_1} \int_{X_2} \chi_{A_1}(x_1)\chi_{A_2}(x_2) f_1(x_1)f_2(x_2) \, d\mu_2(x_2) \, d\mu_1(x_1)
\]

\[
= \int_{X_1} \chi_{A_1}(x_1) \left( \int_{X_2} \chi_{A_2}(x_2) f_1(x_1)f_2(x_2) \, d\mu_2(x_2) \right) \, d\mu_1(x_1)
\]

\[
= \int_{X_1} \chi_{A_1}(x_1) f_1(x_1) \left( \int_{X_2} \chi_{A_2}(x_2) f_2(x_2) \, d\mu_2(x_2) \right) \, d\mu_1(x_1)
\]

\[
= \int_{X_1} \chi_{A_1}(x_1) f_1(x_1) \, d\mu_1(x_1)
\]

\[
= \nu_1(A_1) \nu_2(A_2).
\]

Thus we have

\[
\lambda(A_1 \times A_2) = \nu_1(A_1) \nu_2(A_2) = (\nu_1 \times \nu_2)(A_1 \times A_2).
\]

for all measurable rectangles \( A_1 \times A_2 \). Since our measures are \( \sigma \)-finite, this shows that \( \lambda = \nu_1 \times \nu_2 \), see the remark at the bottom of page 64 in the text. Comparing (*) and (**) and using the uniqueness in (*) then shows that \( f(x_1, x_2) = f_1(x_1)f_2(x_2) \)

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**Problem 6.** [Problem 16, page 92] Suppose that \( \mu, \nu \) are measures on \((X, M)\) with \( \nu \ll \mu \) and let \( \lambda = \mu + \nu \). If \( f = d\nu/d\lambda \), then \( 0 \leq f < 1 \) \( \mu \)-a.e. and \( d\nu/d\mu = f/(1-f) \).

**Answer:**

It’s clear that \( \nu \ll \lambda \), so \( f = d\nu/d\lambda \) makes sense.

We want to show that \( 0 \leq f < 1 \) \( \mu \)-a.e. Suppose, for a contradiction, that this is a set \( E \) with \( \mu(E) > 0 \) and \( f \geq 1 \) on \( E \). On the one hand, we have \( \nu(E) < \nu(E) + \mu(E) = \lambda(E) \). On the other hand, \( f \chi_E \geq \chi_E \), so

\[
\nu(E) = \int_E f \, d\lambda = \int_E \chi_E f \, d\lambda \geq \int \chi_E d\lambda = \lambda(E),
\]

a contradiction.

If \( E \) is measurable, we have

\[
\int_E 1 \, d\nu = \nu(E) = \int_E f \, d\lambda = \int_E f \, d\mu + \int_E f \, d\nu,
\]

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which yields
\[ \int_E (1 - f) \, d\nu = \int_E f \, d\mu, \]
or, in other words,
\[ \int \chi_E (1 - f) \, d\nu = \int \chi_E f \, d\mu. \]

By the usual procedure, we can extend this equation from characteristic functions to nonnegative functions. Thus, we have
\[ \int g(1 - f) \, d\nu = \int g f \, d\mu \]
for all nonnegative measurable functions \( g \). Hence, for any measurable set \( E \), we have
\[ \int_E g(1 - f) \, d\nu = \int_E g f \, d\mu \]
for all nonnegative measurable functions \( g \).

Since \( 0 \leq f < 1 \) \( \mu \)-a.e. the function \( 1/(1 - f) \) is defined and nonnegative \( \mu \)-a.e. Since \( \nu \ll \mu \), \( 1/(1 - f) \) is also defined and nonnegative \( \nu \)-a.e. Thus, we can set \( g = 1/(1 - f) \) in (\( \star \)). This gives
\[ \nu(E) = \int_E \frac{f}{1 - f} \, d\mu, \]
for all measurable \( E \), whence \( d\nu/d\mu = f/(1 - f) \).

**Problem 7.** [Problem 18, page 94] Prove Proposition 3.13c.

In other words, suppose that \( \nu \) is a complex measure on \((X, \mathcal{M})\). Then \( L^1(\nu) = L^1(|\nu|) \) and if \( f \in L^1(\nu) \),
\[ \left| \int f \, d\nu \right| \leq \int |f| \, d|\nu|. \]

**Answer:**
I seem to have been very confused the day I tried to do this in class!

Let \( z = x + iy \) be a complex number, where \( x, y \) are real. We have
\[ x^2, y^2 \leq x^2 + y^2 = |z|^2 \]
so taking square roots gives \(|x|, |y| \leq |z|\). One the other hand, we have
\[ |z|^2 = x^2 + y^2 = |x|^2 + |y|^2 \leq |x|^2 + 2|x||y| + |y|^2 = |x| + |y|^2, \]
and so \(|z| \leq |x| + |y|\).
Now let's begin the solution of the problem. By the definition on page 93 of the text, $L^1(\nu)$ is defined to be $L^1(\nu_r) \cap L^1(\nu_i)$ and for $f \in L^1(\nu)$, we define

$$\int f \, d\nu = \int f \, d\nu_r + i \int f \, d\nu_i.$$

On the other hand, we know from Exercise 3 on page 88 of the text (one of the problems in this set) that for a signed measure $\lambda$, $L^1(\lambda) = L^1(|\lambda|)$ and for $f \in L^1(\lambda)$,

$$\left| \int f \, d\lambda \right| \leq \int |f| \, |d\lambda|.$$

Hence in our present situation, we know $L^1(\nu) = L^1(|\nu_r|) \cap L^1(|\nu_i|)$.

Let's recall the reasoning of part b. of Proposition 3.13. We can find some finite positive measure $\mu$ such that $\nu_r$ and $\nu_i$ are absolutely continuous with respect to $\mu$ ($\mu = |\nu_r| + |\nu_i|$ will do). Then, by the Lebesgue-Radon-Nikodym Theorem, there are real-valued measurable functions $f$ and $g$ such that $d\nu_r = fd\mu$ and $d\nu_i = gd\mu$. Then, we have

\begin{align*}
\nu(E) &= \int_E h \, d\mu \\
\end{align*}

where $h = f + ig$. From this equation, we have $d|\nu| = |h| \, d\mu$, by definition. Since $|f|, |g| \leq |h|$ we have

\begin{align*}
|\nu_r|(E) &= \int_E |f| \, d\mu \leq \int_E |h| \, d\mu = |\nu|(E) \\
|\nu_i|(E) &= \int_E |g| \, d\mu \leq \int_E |h| \, d\mu = |\nu|(E)
\end{align*}

so we certainly have $\nu_r, \nu_i \ll |\nu|$. Thus, once again, we can find real-valued functions $\varphi$ and $\eta$ such that $d\nu_r = \varphi d|\nu|$ and $d\nu_i = \eta d|\nu|$. If $\varphi = \psi + i\eta$, then

$$\nu(E) = \int_E \varphi \, d|\nu|.$$

But then, comparing with $(\ast)$, we have

$$\int_E \varphi |h| \, d\mu = \int_E h \, d\mu.$$

By the uniqueness part of the Lebesgue-Radon-Nikodym Theorem, $\varphi |h| = h$, $\mu$-a.e., and hence $|\nu|$-a.e. On the other hand if $Z$ is the set where $h = 0$, then

$$|\nu|(Z) = \int_Z |h| \, d\mu = \int_Z 0 \, d\mu = 0$$

so $h \neq 0$, $|\nu|$-a.e. This shows that $|\psi| = 1$, $|\nu|$-a.e., so we can conclude that $\varphi, \eta \leq |\varphi| = 1$, $|\nu|$-a.e., and $1 = |\varphi| \leq |\varphi| + |\eta|$, $|\nu|$-a.e. Finally, we have $d|\nu_r| = |\varphi| \, d|\nu|$ and $d|\nu_i| = |\eta| \, d|\nu|$.
Now, suppose that $f \in L^1(\nu)$. Then

$$\int |f| d\nu < \infty$$

Since $|f| \psi \leq |f|, \nu$-a.e., we have

$$\int |f| |\psi| d\nu \leq \int |f| d\nu < \infty$$

but the integral on the left is

$$\int |f| d\nu$$

so $f \in L^1(\nu)$. Similarly, $f \in L^1(\nu_i)$.

Conversely, suppose that $f \in L^1(\nu_r) \cap L^1(\nu_i)$ Then we have

$$\int |f| |\psi| d\nu = \int |f| d\nu_r < \infty$$

$$\int |f| |\eta| d\nu = \int |f| d\nu_i < \infty.$$ and so we have

$$\int |f| |\psi| + |\eta| |d\nu| < \infty.$$

However, we have $1 \leq |\psi| + |\eta|, \nu$-a.e., so

$$\int |f| d\nu \leq \int |f| |\psi| + |\eta| |d\nu| < \infty.$$ and so $f \in L^1(\nu)$.

Finally, suppose that $f \in L^1(\nu) = L^1(\nu)$, we then have

$$\left| \int f \, d\nu \right| = \left| \int f \, d\nu_r + i \int f \, d\nu_i \right|$$

by definition

$$= \left| \int f \psi \, d\nu + i \int f \eta \, d\nu \right|$$

$$= \left| \int f (\psi + i \eta) \, d\nu \right|$$

$$= \left| \int f \varphi \, d\nu \right|$$

$$\leq \int |f| |\varphi| \, d\nu$$

$$= \int |f| \, d\nu,$$

since $|\varphi| = 1, \nu$-a.e. This completes the proof.
Problem 8. [Problem 20, page 94] If \( \nu \) is a complex measure on \((X, \mathcal{M})\) and \( \nu(X) = |\nu|(X) \), then \( \nu = |\nu| \).

Answer:

Let \( f = d\nu/d|\nu| \), so we have

\[
(*) \quad \nu(E) = \int_E f \, d|\nu|
\]

for all measurable sets \( E \), and we know from Proposition 3.13 on page 94 of the text that \(|f| = 1 \) \( |\nu| \)-a.e. We may as well suppose that \(|f| = 1 \) everywhere.

We can write the complex function \( f \) as \( f = g + ih \), where \( g \) and \( h \) are real-valued. We can also write \( \nu = \nu_r + i\nu_i \) for finite signed measures \( \nu_r \) and \( \nu_i \).

Thus, we have

\[
\nu_r(E) + i\nu_i(E) = \nu(E) = \int_E f \, d|\nu| = \int_E g \, d|\nu| + i \int_E h \, d|\nu|.
\]

Comparing real and imaginary parts, we get

\[
\nu_r(E) = \int_E g \, d|\nu|
\]

\[
\nu_i(E) = \int_E h \, d|\nu|.
\]

By hypothesis, we have \( \nu(X) = |\nu|(X) \), so

\[
\int_X g \, d|\nu| + i \int_X h \, d|\nu| = |\nu|(X).
\]

Since the right-hand side is real, the imaginary part of the left-hand side must be zero, and we have

\[
\int_X g \, d|\nu| = |\nu|(X) = \int_X d|\nu|,
\]

and so

\[
(**) \quad 0 = \int_X (1 - g) \, d|\nu|.
\]

Since \( g \) is the real part of \( f \), we have

\[
g \leq |g| \leq |g + ih| = |f| = 1
\]

so \( 1 - g \geq 0 \). But then \((**)\) shows that \( 1 - g = 0 \) a.e., so \( g = 1 \) a.e.

We then have \( 1 = |f|^2 = g^2 + h^2 = 1 + h^2 \) a.e., so \( h = 0 \) a.e. and hence \( f = g = 1 \) a.e. Putting this in \((*)\) shows that \( \nu = |\nu| \).
Problem 9. [Problem 20, page 94] Let \( \nu \) be a complex measure on \( (X, \mathcal{M}) \).

If \( E \in \mathcal{M} \), define

(A) \( \mu_1(E) = \sup \left\{ \sum_{j=1}^{n} |\nu(E_j)| \mid n \in \mathbb{N}, E_1, \ldots, E_n \text{ disjoint}, E = \bigcup_{j=1}^{n} E_j \right\} \),

(B) \( \mu_2(E) = \sup \left\{ \sum_{j=1}^{\infty} |\nu(E_j)| \mid E_1, E_2, \ldots \text{ disjoint}, E = \bigcup_{j=1}^{\infty} E_j \right\} \),

(C) \( \mu_3(E) = \sup \left\{ \left| \int_{E} f \, d\nu \right| \mid |f| \leq 1 \right\} \).

Then \( \mu_1 = \mu_2 = \mu_3 = |\nu| \).

Answer:
Although the author didn’t explicitly say so, the sets \( E_j \) in (A) and (B) should, of course, be measurable and the functions \( f \) in (C) should be measurable. Also note that the book has a typo in (C) (\( d\mu \) instead of \( d\nu \)).

As suggested in the book, we first prove that \( \mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_1 \) and then that \( \mu_3 = |\nu| \).

We have \( \mu_1 \leq \mu_2 \), because the sums in (A) are among the sums in (B): given a finite sequence \( E_1, \ldots, E_n \) of disjoint sets whose union is \( E \), just set \( E_j = \emptyset \) for \( j > n \). Then \( E_1, E_2, \ldots \) is an infinite sequence of disjoint sets whose union is \( E \) and

\[
\sum_{j=1}^{n} |\nu(E_j)| = \sum_{j=1}^{\infty} |\nu(E_j)|.
\]

Thus, the set of numbers in (A) is a subset of the set of numbers in (B). Since the sup over a larger set is larger, \( \mu_1 \leq \mu_2 \).

To see that \( \mu_2 \leq \mu_3 \), let \( E_1, E_2, \ldots \) be a sequence of disjoint sets whose union is \( E \). Now each \( \nu(E_j) \) is a complex number, so there is a complex number \( \omega_j \) such that \( |\omega_j| = 1 \) and \( \omega_j \nu(E_j) = |\nu(E_j)| \). Indeed, we can say

\[
\omega_j = \begin{cases} 
\frac{|\nu(E_j)|}{\nu(E_j)} & \nu(E_j) \neq 0 \\
1 & \nu(E_j) = 0.
\end{cases}
\]

Consider the function \( f \) defined by

\[
f = \sum_{j=1}^{\infty} \omega_j \chi_{E_j}.
\]

If \( x \notin E \), \( f(x) = 0 \). If \( x \in E \), it is in exactly one \( E_j \) and then \( |f(x)| = |\omega_j| = 1 \).

Thus, \( |f| \leq 1 \) and the partial sums

\[
f_n = \sum_{j=1}^{n} \omega_j \chi_{E_j}
\]
satisfy $|f_n| \leq 1$ by the same reasoning. We then have

$$\left| \int f_n \, d\nu - \int f \, d\nu \right| = \left| \int (f_n - f) \, d\nu \right| \leq \int |f_n - f| \, d|\nu| \to 0$$

by the dominated convergence theorem, since $f_n \to f$ pointwise and $|f_n - f|$ is bounded above by the constant function 2, which is integrable with respect to the finite measure $|\nu|$. We then have

$$\int f \, d\nu = \lim_{n \to \infty} \int f_n \, d\nu$$

$$= \sum_{j=1}^{\infty} \int_{E_j} \omega_j \chi_{E_j} \, d\nu$$

$$= \sum_{j=1}^{\infty} \omega_j \nu(E_j)$$

$$= \sum_{j=1}^{\infty} |\nu(E_j)|.$$

since $\int_E f \, d\nu$ is positive, we have

$$\left| \int_E f \, d\nu \right| = \sum_{j=1}^{\infty} |\nu(E_j)|.$$

Thus, the set of numbers in (B) is a subset of the set in (C), so $\mu_2 \leq \mu_3$.

Now we want to show that $\mu_3 \leq \mu_1$. First, we use Proposition 3.13 (and Exercise 18) to show that $\mu_3$ is finite. If $|f| \leq 1$, then

$$\int |f| \, d|\nu| \leq \int 1 \, d|\nu| = |\nu|(X) < \infty,$$

so $f \in L^1(|\nu|) = L^1(\nu)$ and

$$\left| \int_E f \, d\nu \right| \leq \int_E |f| \, d|\nu| \leq \int_E 1 \, d|\nu| = |\nu|(E).$$

Thus, $|\nu|(E)$ is an upper bound for the set of numbers in (C), so $\mu_3(E) \leq |\nu|(E) \leq |\nu|(X) < \infty$.

To show that $\mu_3 \leq \mu_1$, let $\varepsilon > 0$ be given. Then we can find some $f$ with $|f| \leq 1$ such that

$$\mu_3(E) - \varepsilon < \left| \int_E f \, d\nu \right|,$$

which implies

$$(*) \quad \mu_3(E) \leq \left| \int_E f \, d\nu \right| + \varepsilon.$$
We approximate $f$ by a simple function as follows. Let

$$D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$$

be the closed unit disk in the complex plane. Since $|f| \leq 1$, all the values of $f$ are in $D$. The collection of balls

$$\{ B(\varepsilon, z) \mid z \in D \}$$

is an open cover of $D$. Since $D$ is compact, there is a finite subcover, say

$$\{ B(\varepsilon, z_j) \mid j = 1, \ldots, m \}.$$  

Define $B_j = f^{-1}(B(\varepsilon, z_j)) \subseteq X$, which is measurable since $f$ is measurable. The union of the sets $B_j$ is $X$. The sets $B_j$ won’t be disjoint, but we can use the usual trick and define

$$A_1 = B_1,$$

$$A_j = B_j \setminus \bigcup_{k=1}^{j-1} B_k$$

to get a disjoint collection of sets with $A_j \subseteq B_j$ and $\bigcup_j A_j = X$. Define a simple function $\varphi$ by

$$\varphi = \sum_{j=1}^{m} z_j \chi_{A_j}.$$  

This function takes on the values $z_j$, which are in $D$, so $|\varphi| \leq 1$. If $x \in X$ it is in exactly one of the sets $A_j$. But then $f(x) \in B(\varepsilon, z_j)$, so

$$|f(x) - \varphi(x)| = |f(x) - z_j| < \varepsilon.$$  

Thus, $|f - \varphi| < \varepsilon$. We then have

$$\left| \int_{E} f \, d\nu \right| - \left| \int_{E} \varphi \, d\nu \right| \leq \left| \int_{E} f \, d\nu - \int_{E} \varphi \, d\nu \right| \leq \int_{E} |f - \varphi| \, d\nu \leq \int_{E} \varepsilon \, d\nu \leq \varepsilon |\nu|(E).$$  

Thus, we have

$$\left| \int_{E} f \, d\nu \right| \leq \left| \int_{E} \varphi \, d\nu \right| + \varepsilon |\nu|(E).$$
Substituting this is (∗) gives

\[ \mu_3(E) \leq \left| \int_E \varphi \, d\nu \right| + \varepsilon + \varepsilon |\nu|(E). \]

Now define \( E_j = A_j \cap E, \ j = 1, \ldots, m \). Then the \( E_j \)'s are a finite sequence of disjoint sets whose union is \( E \) and we have

\[
\left| \int_E \varphi \, d\nu \right| = \left| \int_E \left[ \sum_{j=1}^m z_j \chi_{A_j} \right] \, d\nu \right|
\]
\[
= \left| \sum_{j=1}^m z_j \int_E \chi_{A_j} \, d\nu \right|
\]
\[
= \left| \sum_{j=1}^m z_j \int_E \chi_E \chi_{A_j} \, d\nu \right|
\]
\[
= \left| \sum_{j=1}^m z_j \int_E \chi_{E_j} \, d\nu \right|
\]
\[
= \left| \sum_{j=1}^m z_j \nu(E_j) \right|
\]
\[
\leq \sum_{j=1}^m |z_j| |\nu(E_j)|
\]
\[
\leq \sum_{j=1}^m |\nu(E_j)|
\]
\[
\leq \mu_1(E).
\]

Substituting this into (**) gives

\[ \mu_3(E) \leq \mu_1(E) + [1 + |\nu|(E)]\varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, we conclude that \( \mu_3(E) \leq \mu_1(E) \).

We’ve now shown that \( \mu_1 = \mu_2 = \mu_3 \). The last step is to show that \( \mu_3 = |\nu| \).

We’ve already shown above that \( \mu_3(E) \leq |\nu|(E) \). To get the reverse inequality, let \( g = d\nu/d|\nu| \). We know that \( |g| = 1, |\nu|-a.e. \), and we may as well assume that \( |g| = 1 \) everywhere (by modifying it on a set of measure zero). Then the conjugate \( \bar{g} \) of \( g \) satisfies \( |\bar{g}| = 1 \leq 1 \) and so is one of the functions in the definition of \( \mu_3 \), so

\[ \left| \int_E \bar{g} \, d\nu \right| \leq \mu_3(E). \]
But, from the definition of $g$,

\[
\left| \int_E g \, d\nu \right| = \left| \int_E \bar{g} g \, d|\nu| \right|
\]
\[
= \left| \int_E |g|^2 \, d|\nu| \right|
\]
\[
= \left| \int_E 1 \, d|\nu| \right|
\]
\[
= |\nu|(E).
\]

Thus, $|\nu|(E) \leq \mu_3(E)$, so $\mu_3 = |\nu|$, and we’re done.