## Type Of Securities

Terminologies (roughly speaking):
Bond: Issuer borrows money for a finite time period and payback periodically through coupons and/or payback the rest at the end of term.
Preferred stock: Issuer borrows money for indefinite time period and payback periodically.
Common stock: Issuer borrows money for indefinite time period and pay back whenever and whatever they prefer.

## Example (Exercise 6.1)

Find the price which should be paid for a zero coupon bond that matures for $\$ 1,000$ in 10 years to yield:
a) $10 \%$ effective.
b) $9 \%$ effective.
c) Thus, a $10 \%$ reduction in the yield rate causes the price to increase by what percentage?

## Example (Exercise 6.2)

A 10-year accumulation bond with an initial par value of $\$ 1,000$ earns interest of $8 \%$ compounded semiannually. Find the price to yield an investor $10 \%$ effective.

## Price Of A Bond I

## Notations:

$P=$ the price of a bond.
$F=$ the par value or face value of a bond. It is the value printed on the face of the bond, its percentage determines the value of the coupons (" $x \%$ bond of par value $F$ " means that the coupon value per year is $x \%$ of $F$ ), and is often the redemption value at the maturity date.
$C=$ the redemption value of $a$ bond. Always assume $C=F$ unless stated otherwise.
$r=$ the coupon rate of a bond. The amount of the coupon is
$F \times r$.
Fr amount of the coupon. Total $F r$ per year $=F \times r$;
$g=$ the modified coupon rate of a bond. The amount of the coupon per year $=C \times g$.

## Price Of A Bond II

$i=$ the yield rate of a bond, also called yield to maturity rate. It is the interest rate eared by the whole investment (coupons and redemption).
$n=$ the number of coupon periods.
$K=$ the preset value of $C$.
$G=$ the base amount of a bond. $G$ generates a perpetuity of coupon amounts: $G=\frac{F r}{i}$.

## Price Of A Bond III

## Basic Formula

Present value of a bond $=$ Present value of coupons (an annuity) + Present value of redemption value:

$$
P=F r a_{\bar{n}}+v^{n} C .
$$

## Example (Exercise 6.4)

A 10-year $\$ 100$ par value bond bearing a $10 \%$ coupon rate payable semiannually and redeemable at $\$ 105$ is bought to yield $8 \%$ convertible semiannually. Find the price.

## Example (Exercise 6.5)

Two $\$ 1000$ bonds redeemable at par at the end of the same period are bought to yield $4 \%$ convertible semiannually. One bond costs $\$ 1136.78$ and has a coupon rate of $5 \%$ payable semiannually. The other bond has a coupon rate of $2 \frac{1}{2} \%$ payable semiannually. Find the price of the second bond.

## Example (Exercise 6.6)

A $\$ 1000$ bond with a coupon rate of $9 \%$ payable semiannually is redeemable after an unspecified number of years at $\$ 1125$. The bond is bought to yield $10 \%$ convertible semiannually, if the present value of the redemption value is $\$ 225$ at this yield rate, find the purchase price.

## Example (Exercise 6.7)

A $\$ 1000$ par value $n$-year bond maturing at par with $\$ 100$ annual coupons is purchased for $\$ 1110$. If $K=\$ 450$, find the base amount $G$.

## Example (Exercise 6.8)

An investor owns a $\$ 1000$ par value $10 \%$ bond with semiannual coupons. The bond will mature at par at the end of 10 years. The investor decides that an 8 -year bond would be preferable. Current yield rates are $7 \%$ convertible semiannually. The investor uses the proceeds from the sale of the $10 \%$ bond to purchase a $6 \%$ bond with semiannual coupons, maturing at par at the end of 8 years. Find the par value of the 8 -year bond.

## Example (Exercise 6.9)

An n-year $\$ 1000$ par value bond matures at par and has a coupon rate of $12 \%$ convertible semiannually. It is bought at a price to yield $10 \%$ convertible semiannually. If the term of the bond is doubled, the price will increase by $\$ 50$. Find the price of the $n$-year bond.

## More Terminologies

$$
\begin{gathered}
\text { Nominal yield }=\frac{\text { total coupon values per year }}{\text { F or C }} \\
\text { Current yield }=\frac{\text { total coupon values per year }}{P} \\
\text { Yield to maturity }=i
\end{gathered}
$$

## Example (Exercise 6.10)

A $\$ 1000$ par value 10 -year bond with coupons at $8.4 \%$ payable semiannually, which will be redeemed at $\$ 1050$. The bond is bought to yield $10 \%$ convertible semiannually. Determine the following:
a) Nominal yield, based on the par value.
b) Nominal yield, based on the redemption value.
c) Current yield.
d) Yield to maturity.

## Premium And Discount I

The coupon value is given by $F r=F \times r=C \times g$, and the price of a bond can be written as

$$
\begin{aligned}
P & =F r a_{\bar{n}}+v^{n} C=C g a_{\bar{n}}+C\left(v^{n}-1\right)+C \\
& =C g a_{\bar{n}}-C i a_{\bar{n}}+C=C(g-i) a_{\bar{n}}+C
\end{aligned}
$$

If $g>i$, i.e. the coupons pay more than what the interest rate calculated, then $P>C$, and the bond is said to sell at "premium" and $P-C$ is called premium.
If $g<i$, i.e. the coupons pay less than what the interest rate calculated, then $P<C$, and the bond is said to sell at "discount" and $C-P$ is called discount.

## Premium And Discount II

Similar to mortgages, we have also the "bond amortization schedule". Recall for mortgages, the loan balance equals the present value of all future payments. The corresponding term for bond is the book value, which equals the present value of all future coupons and redemption values. The difference of current and previous book values is call amortization of premium for premium bond, and is called accumulation of discount for discount bond. The interest earned equals the sum of coupon and current book values minus the previous book value.

## Premium And Discount III

## Bond Amortization Schedule

|  | Premium $(P>C)$ | Discount $(P<C)$ |
| :---: | :---: | :---: |
| Book Value | $B_{j}=F r a \frac{n}{n-j}+v^{n-j} C$ |  |
| $\left\|B_{j}-B_{j-1}\right\|$ | Amortization <br> of Premium | Accumulation <br> of Discount |
| Interest Earned | $F r+B_{j}-B_{j-1}$ |  |

## Premium And Discount IV

If $g=i$, then $P=C$, and $F r=C g=P i$, i.e. the coupon is simply the interest that the bond issuer (borrower) pays to the buyer (lender) for borrowing $P$. At the end of term, the issuer pays off the loan $P$.

For the premium bond, $P=C+(P-C)$, which means that the issuer borrows also premium $P-C$ in addition to $C$, and the coupon includes interest and amortization of the premium. To calculate the amortization schedule, we first find the difference of the book values (amortization of premium). The coupon minus the amortization of the premium is the interest payment.

## Premium And Discount V

For the discount bond, $P=C-(C-P)$, which means that the discount $C-P$ is deducted from the total amount $C$ of the loan, and the coupon plus part of the discount (accumulation of discount) is the interest payment. To calculate the amortization schedule, we first find the difference of the book values (accumulation of discount). The coupon plus the accumulation of discount is the interest payment.

## Premium And Discount VI

Straight Line Method (An approximation formula) By using the point-slop formula, we have the straight line connecting the points $(0, P)$ and $(n, C)$ :

$$
y-P=\frac{C-P}{n-0}(x-0)
$$

Straight Line Approximation

$$
B_{j} \approx P+\frac{C-P}{n} j
$$

## Example (A)

a) Complete the amortization schedule for a $\$ 1000$ two-year bond with $10 \%$ coupons paid semiannually and yield rate $8 \%$ convertible semiannually.
b) Find the book values by the straight line method.
$F=C=1000, n=2(2)=4$, coupon $=50$. Since the coupon rate is higher than the interest rate, it is a premium bond.
We first calculate the book values: $B_{j}=50 a \overline{4-j}+1000 v^{4-j}$

| N | I | PV | PMT | FV | P/Y | C/Y | E/B? |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $4-\mathrm{j}$ | 8 |  | -50 | -1000 | 2 | 2 | END |
|  |  | $\Downarrow$ |  |  |  |  |  |
|  |  | $B_{j}$ |  |  |  |  |  |

$B_{0}=1036.30, B_{1}=1027.75, B_{2}=1018.86, B_{3}=1009.62$,
$B 4=C=1000$.
$A_{1}=B_{0}-B_{1}=8.55, A_{2}=B_{1}-B_{2}=8.89$,
$A_{3}=B_{2}-B_{3}=9.24, A_{4}=B_{3}-B_{4}=9.2$.
$I_{1}=50-A_{1}=41.45, I_{2}=50-A_{2}=41.11$,
$I_{3}=50-A_{3}=40.76, I_{4}=50-A_{4}=40.38$.
Finally the straight line method gives us $B_{j} \approx B_{0}+\frac{1000-B_{0}}{4} j$.

|  | Coupon | Interest <br> Earned | Amortization <br> of Premium | Book <br> Value | Book Value <br> Staightline |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 |  |  |  | 1036.30 | 1036.30 |
| 1 | 50 | 41.45 | 8.55 | 1027.75 | 1027.22 |
| 2 | 50 | 41.11 | 8.89 | 1018.86 | 1018.15 |
| 3 | 50 | 40.76 | 9.24 | 1009.62 | 1009.07 |
| 4 | 50 | 40.38 | 9.62 | 1000.00 | 1000.00 |

## Example (B)

a) Complete the amortization schedule for a $\$ 1000$ two-year bond with $6 \%$ coupons paid semiannually and yield rate $8 \%$ convertible semiannually.
b) Find the book values by the straight line method.
$F=C=1000, n=2(2)=4$, coupon $=30$. Since the coupon rate is lower than the interest rate, it is a discount bond.

We first calculate the book values: $B_{j}=30 a \overline{4-j}+1000 v^{4-j}$

| N | I | PV | PMT | FV | $\mathrm{P} / \mathrm{Y}$ | C/Y | E/B? |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4-j | 8 |  | -30 | -1000 | 2 | 2 | END |
|  |  | $\begin{aligned} & \Downarrow \\ & B_{j} \end{aligned}$ |  |  |  |  |  |

$B 4=C=1000$.
$A_{1}=B_{1}-B_{0}=8.55, A_{2}=B_{2}-B_{1}=8.89$,
$A_{3}=B_{3}-B_{2}=9.24, A_{4}=B_{4}-B_{3}=9.62$.
$I_{1}=30+A_{1}=38.55, I_{2}=30+A_{2}=38.89$,
$I_{3}=30+A_{3}=39.24, I_{4}=30+A_{4}=39.62$.
Finally the straight line method gives us $B_{0}+\frac{1000-B_{0}}{4} j$.

|  | Coupon | Interest <br> Earned | Accumulation <br> of Discount | Book <br> Value | Book Value <br> Staightline |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 |  |  |  | 963.70 | 963.70 |
| 1 | 30 | 38.55 | 8.55 | 972.25 | 972.78 |
| 2 | 30 | 38.89 | 8.89 | 981.14 | 981.85 |
| 3 | 30 | 39.24 | 9.24 | 990.38 | 990.93 |
| 4 | 30 | 39.62 | 9.62 | 1000.00 | 1000.00 |

The price $P$ can also be expressed as

$$
P=F r a_{\bar{n}}+C v^{n}=C g a_{\bar{n}}+C\left(v^{n}-1\right)+C=C(g-i) a_{\bar{n}}+C .
$$

So,
For a premium bond

$$
\text { Premium }=P-C=C(g-i) a_{\bar{n}} .
$$

For a discount bond
Discount $=C-P=C(i-g) a_{\bar{n}}$.

Similar to annuities, for a premium bond, the amount of amortization of premium in the $j$ th coupon is given by

$$
C(g-i) v^{n-j+1} .
$$

Similarly, for a discount bond, the amount of accumulation of discount in the $j$ th coupon is given by

$$
C(i-g) v^{n-j+1}
$$

Recall that for annuity we have $p r_{j}=(p m) v^{n-j+1}$ and $p r_{j}=p r_{i} u^{j-i}$.

For a premium bond, the amount of amortization of premium in the $j$ th coupon is given by

$$
p r_{j}=C(g-i) v^{n-j+1} \quad \text { and } \quad p r_{j}=p r_{i} u^{j-i} .
$$

For a discount bond, the amount of accumulation of discount in the $j$ th coupon is given by

$$
d i_{j}=C(i-g) v^{n-j+1} \quad \text { and } \quad d i_{j}=d i_{i} u^{j-i} .
$$

Note that unless stated otherwise, we always assume $F=C$ and $g=r$.

## Example (Exercise 6.11)

For a $\$ 1$ bond the coupon rate is $150 \%$ of the yield rate and the premium is $p$. For another $\$ 1$ bond with the same number of coupons and the same yield rate, the coupon rate is $75 \%$ of the yield rate. Express the price of the second bond as a function of $p$.

## Example (Exercise 6.12)

For a certain period a bond amortization schedule shows that the amount for amortization of premium is $\$ 5$ and that the required interest is $75 \%$ of the coupon. Find the amount of the coupon.

## Example (Exercise 6.13)

A 10-year bond with semiannual coupons is bought at a discount to yield $9 \%$ convertible semiannually. If the amount for accumulation of discount in the 19th coupon is $\$ 8$, find the total amount for accumulation of discount during the first four years in the bond amortization schedule.

## Example (Exercise 6.14)

A $\$ 1000$ par value five-year bond with a coupon rate of $10 \%$ payable semiannually and redeemable at par is bought to yield $12 \%$ convertible semiannually. Find the total of the interest paid column in the bond amortization schedule.

## Example (Exercise 6.15)

You are given:
(i) A 10-year $8 \%$ semiannual coupon bond is purchased at a discount of $X$.
(ii) A 10-year 9\% semiannual coupon bond is purchased at a premium of $Y$.
(iii) A 10-year 10\% semiannual coupon bond is purchased at a premium of $2 X$.
(iv) All bonds were purchased at the same yield rate and have the same par value.
Calculate $Y$ in terms of $X$.

## Valuation Between Coupon Payment Dates I

There are two ways to compute the book value of a bond between the coupon periods at the time $j+k, 0<k<1$. The flat price, denoted by $B_{j+k}^{f}$, is the present value of all future coupons and redemption values. $B_{j+k}^{f}$ can be computed by accumulating the book value at $t=j$ to the time $t=j+k$,

$$
B_{j+k}^{f}=B_{j} u^{k}=\left(F r a_{\overline{n-j}}+C v^{n-j}\right) u^{k}
$$

The flat price is also called "price-plus-accrued", "full price", or "dirty price".

## Valuation Between Coupon Payment Dates II

The market price, denoted by $B_{j+k}^{m}$, is to let $t=j+k$ in the formula of $B_{t}$,

$$
B_{j+k}^{m}=F r a \overline{n-j-k}+C v^{n-j-k}
$$

The market price is also called "clean price".

## Valuation Between Coupon Payment Dates III

The flat price is higher than the market price, and the difference is called accrued coupon, and is denoted by $F r_{k}$,

$$
\begin{aligned}
F r_{k} & =B_{j+k}^{f}-B_{j+k}^{m} \\
& =\left(F r \frac{1-v^{n-j}}{i}+C v^{n-j}\right) u^{k}-\left(F r \frac{1-v^{n-j-k}}{i}+C v^{n-j-k}\right) \\
& =\operatorname{Fr}\left(\frac{u^{k}-v^{n-j-k}}{i}-\frac{1-v^{n-j-k}}{i}\right) \\
& =\operatorname{Fr} \frac{u^{k}-1}{i}=F r s_{k}
\end{aligned}
$$

## Valuation Between Coupon Payment Dates IV

One can also use linear approximation of $u^{k}$

$$
u^{k}=(1+i)^{k} \approx 1+k i
$$

to compute $B_{j+k}^{f}$ and/or $F r_{k}$,

$$
B_{j+k}^{f} \approx B_{j}(1+k i), \quad F r_{k} \approx F r \frac{1+k i-1}{i}=k F r .
$$

## Valuation Between Coupon Payment Dates V

So there are 3 methods to compute the flat price, accrued coupon, and market price:

|  | $B_{j+k}^{f}$ | $F r_{k}$ | $B_{j+k}^{m}$ |
| :---: | :---: | :---: | :---: |
| theoretical | $B_{j} u^{j}$ | $F r \frac{u^{k}-1}{i}$ | $B_{j+k}^{f}-F r_{k}$ |
| practical | $B_{j}(1+i k)$ | $k F r$ | $B_{j+k}^{f}-F r_{k}$ |
| semi-theoretical | $B_{j} u^{j}$ | $k F r$ | $B_{j+k}^{f}-F r_{k}$ |

## Example (Exercise 6.18)

Find the flat price, accrued interest, and market price (book value) two months after purchase for a $\$ 1000$ two-year bond with $8 \%$ coupons paid semiannually and yield rate $10 \%$ convertible semiannually. Use all three methods.

## Example (Exercise 6.19)

A $\$ 1000$ bond with semiannual coupons at $r=6 \%$ matures at par on October 15, $Z+15$. The bond is purchased on June 28, $Z$ to yield the investor $i^{(2)}=7 \%$. What is the purchase price? Assume simple interest between bond coupon dates and use an exact day count.

## Determination Of Yield Rates I

By a financial calculator:

$$
N=n, \quad P V=P, \quad F V=-C, \quad P M T=-F r
$$

and

$$
\begin{gathered}
P / Y=C / Y=1 \quad \text { for annual coupon } \\
P / Y=C / Y=2 \quad \text { for semiannual coupon } \\
P / Y=C / Y=4 \quad \text { for quarterly coupon }
\end{gathered}
$$

with compound period equals coupon period.

## Determination Of Yield Rates II

There is also an approximated formula (refined version of the bond salesman's method) derived from linearization: the interest per coupon period is given approximately by

$$
i \approx \frac{g-\frac{k}{n}}{1+\frac{n+1}{2 n} k}
$$

where $k=\frac{P-C}{C}$ and $g$ is the modified coupon rate per coupon.

## Example (Exercise 6.20)

A $\$ 100$ par value 12 -year bond with $10 \%$ semiannual coupons is selling for $\$ 110$. Find the yield rate convertible semiannually:
a) Using the exact method.
b) Using the refined version of the bond salesman's method.

## Example (Exercise 6.21)

An investor buys two 20-year bonds, each having semiannual coupons and each maturing at par. For each bond the purchase price produces the same yield rate. One bond has a par value of $\$ 500$ and a coupon of $\$ 45$. The other bond has a par value of $\$ 1000$ and a coupon of $\$ 30$. The dollar amount of premium on the first bond is twice as great as the dollar amount of discount on the second bond. Find the yield rate convertible semiannually.

## Example (Exercise 6.22)

A $\$ 100$ bond with annual coupons is redeemable at par at the end of 15 years. At a purchase price of $\$ 92$ the yield rate is exactly $1 \%$ more than the coupon rate. Find the yield rate on the bond.

## Example (Exercise 6.23)

An n-year $\$ 1000$ par value bond with $4.20 \%$ annual coupons is purchased at a price to yield an annual effective rate of $i$. You are given:
(i) If the annual coupon rate had been $5.25 \%$ instead of $4.20 \%$, the price of the bond would have increased by $\$ 100$.
(ii) At the time of purchase, the present value of all the coupon payments is equal to the present value of the bond's redemption value of $\$ 1000$.
Calculate $i$.

## Callable And Putable Bonds I

A callable bond is a bond for which issuer has an option to redeem (call) early.

A putable bond is a bond for which owner has an option to redeem (put) early.

## Callable And Putable Bonds II

When compute the current value of callable bond, always assume to the best interest of the issuer.

For example, for a premium bond, the issuer pays higher coupon value, so always assume the earliest call date, and for a discount bond, the issuer pays lower coupon value, so always assume the latest call date.

## Callable And Putable Bonds III

When compute the current value of putable bond, always assume to the best interest of the owner.

For example, for a premium bond, the owner receives higher coupon payment, so always assume the latest put date, and for a discount bond, the owner receives lower coupon payment, so always assume the earliest put date.

## Example (Exercise 6.24)

A $\$ 1000$ par value bond has $8 \%$ semiannual coupons and is callable at the end of the 10th through the 15th years at par.
a) Find the price to yield $6 \%$ convertible semiannually.
b) Find the price to yield $10 \%$ convertible semiannually.
c) If the bond in (b) is actually called at the end of 10 years, find the yield rate.
d) If the bond is putable rather than callable, rework (a).
e) If the bond is putable rather than callable, rework (b).

## Example (Exercise 6.25)

A $\$ 1000$ par value $8 \%$ bond with quarterly coupons is callable five years after issue. The bond matures for $\$ 1000$ at the end of ten years and is sold to yield a nominal rate of $6 \%$ convertible quarterly under the assumption that the bond will not be called. Find the redemption value at the end of five years that will provide the purchaser the same yield rate.

## Example (Exercise 6.26)

A $\$ 1000$ par value $4 \%$ bond with semiannual coupons matures at the end of 10 years. The bond is callable at $\$ 1050$ at the ends of years 4 through 6 , at $\$ 1025$ at the ends of years 7 through 9 . and at $\$ 1000$ at the end of year 10 . Find the maximum price that an investor can pay and still be certain of a yield rate of $5 \%$ convertible semiannually.

## Example (Exercise 6.27)

A $\$ 1000$ par value bond with coupons at $9 \%$ payable semiannually was called for $\$ 1100$ prior to maturity. The bond was bought for $\$ 918$ immediately after a coupon payment and was held to call. The nominal yield rate convertible semiannually was $10 \%$. Calculate the number of years the bond was held. Answer to the nearest integer.

## Example (Exercise 6.28)

A $\$ 1000$ par value bond pays annual coupons of $\$ 80$. The bond is redeemable at par in 30 years, but is callable any time from the end of the 10th year at $\$ 1050$. Based on the desired yield rate, an investor calculates the following potential purchase prices $P$ :
(i) Assuming the bond is called at the end of the 10th year, $P=\$ 957$.
(ii) Assuming the bond is held until maturity, $P=\$ 897$.

The investor buys the bond at the highest price that guarantees the desired yield rate regardless of when the bond is called. The investor holds the bond for 20 years. after which time the bond is called. Calculate the annual yield rate the investor earns.

## Other Securities

## Example (Exercise 6.35)

A preferred stock pays a $\$ 10$ dividend at the end of the first year, with each successive annual dividend being 5\% greater than the preceding one. What level annual dividend would be equivalent if $i=12 \%$ ?

## Example (Exercise 6.36)

A common stock pays annual dividends at the end of each year. The earnings per share in the year just ended were \$6. Earnings are assumed to grow $8 \%$ per year in the future. The percentage of earnings paid out as a dividend will be $0 \%$ for the next 5 years and $50 \%$ thereafter. Find the theoretical price of the stock to yield an investor 15\% effective.

## Example (Exercise 6.37)

A common stock is purchased at a price equal to 10 times current earnings. During the next 6 years the stock pays no dividends, but earnings increase $60 \%$. At the end of 6 years the stock is sold at a price equal to 15 times earnings. Find the effective annual yield rate earned on this investment.

## Example (Exercise 6.38)

A $\$ 100$ par value $10 \%$ preferred stock with quarterly dividends is bought to yield $8 \%$ convertible quarterly into perpetuity. However, the preferred stock is actually called at the end of 10 years at par. Find the nominal yield rate convertible quarterly that an investor would actually earn over the 10 -year period.

