Sum of A Geometric Series I

We first find out the sum of n-term geometric series

$$1 + r + r^2 + \cdots + r^{n-1},$$

where r is any number. Multiply it by $1 - r$, we have

$$\begin{array}{rrrrrr}
 & 1 & +r & +r^2 & +\cdots & +r^{n-1} \\
\times & & & & 1 & -r \\
\hline
 & -r & -r^2 & -\cdots & -r^{n-1} & -r^n \\
1 & +r & +r^2 & +\cdots & +r^{n-1} & \\
\hline
1 & & & & & -r^n
\end{array}$$

Sum of A Geometric Series II

$$1 + r + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r},$$

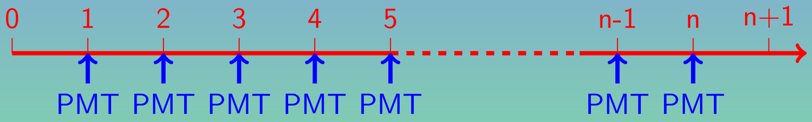
and

$$Ar^k + Ar^{k+1} + \cdots + Ar^m = Ar^k(1 + r + \cdots + r^{m-k}) = Ar^k \frac{1 - r^{m-k+1}}{1 - r}.$$

Annuity I

Definition

An annuity is a series of payments made at equal intervals of time.



Clearly for an annuity of payments 1's

$$\text{Value at the first payment} = 1 + v + v^2 + \cdots + v^{n-1} = \frac{1 - v^n}{1 - v},$$

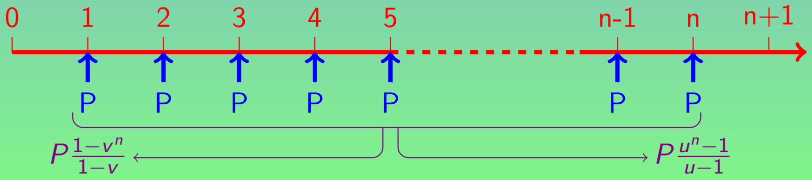
where v is the discount factor per period,

Annuity II

and

$$\begin{aligned}\text{Value at the last payment} &= 1 + u + u^2 + \cdots + u^{n-1} = \frac{1-u^n}{1-u} \\ &= \frac{u^n - 1}{u - 1}\end{aligned}$$

where u is the accumulation factor per period.



Annuity-Immediate I

Definition

Annuity-Immediate: Payments are made at the end of each of the n periods.
The present value at $t = 0$ for payments 1's is denoted by

$$a_{\overline{n}|}$$

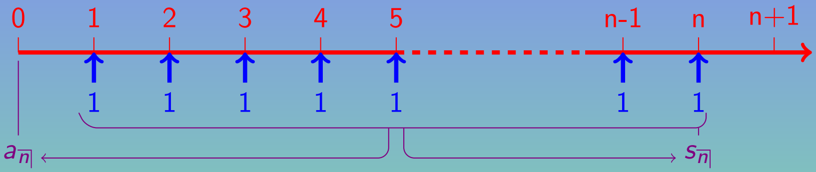
The accumulated value at $t = n$ for payments 1's is denoted by

$$s_{\overline{n}|}$$

Some times people also put the interest rate in the subscript, for example:

$$a_{\overline{n}|0.09}, \qquad s_{\overline{n}|0.085}.$$

Annuity-Immediate II



So

$$a_{\overline{n}|} = v\frac{1-v^n}{1-v} = \frac{1-v^n}{\frac{1}{v}-1} = \frac{1-v^n}{i},$$

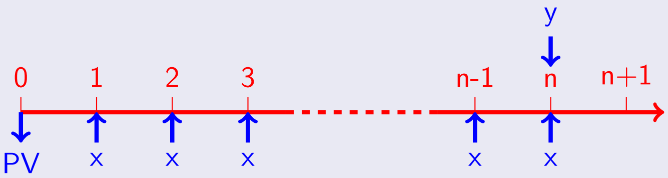
$$s_{\overline{n}|} = \frac{u^n-1}{u-1} = \frac{u^n-1}{i}.$$

Annuity-Immediate III

Use END mode in a financial calculator to:

calculate $PV = xa_{\overline{n}|} + yv^n$, or $PV - xa_{\overline{n}|} - yv^n = 0$,

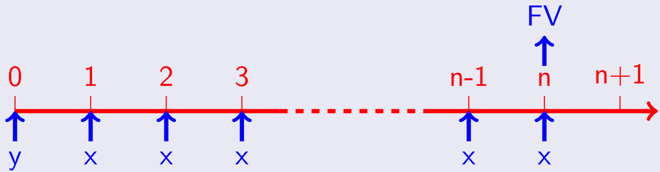
$$\text{input:} \quad PMT = -x, \; FV = -y, \quad \text{output:} \quad PV.$$



Annuity-Immediate IV

calculate $FV = xs_{\overline{n}|} + yu^n$, or $FV - xs_{\overline{n}|} - yu^n = 0$,

$$\text{input:} \quad PMT = -x, \; PV = -y, \quad \text{output:} \quad FV.$$



Example (Exercise 3.1)

A family wishes to accumulate $50,000 in a college education fund at the end of 20 years. If they deposit $1000 in the fund at the end of each of the first 10 years and $1000 + x$ in the fund at the end of each of the second 10 years, find x if the fund earns 7% effective.

Example (Exercise 3.2)

The cash price of an automobile is $\$10,000$. The buyer is willing to finance the purchase at 18% convertible monthly and to make payments of $\$250$ at the end of each month for four years. Find the down payment that will be necessary.

Example (Exercise 3.3)

A sports car enthusiast needs to finance $\$25,000$ of the total purchase price of a new car. A loan is selected having 48 monthly level payments with a lender charging 6% convertible monthly. However, the lender informs the buyer that their policy is not to exceed a \$500 monthly payment on any car Loan. The buyer decides to accept the loan offer with the \$500 payment and then decides to take out a second 12-month loan with a different lender at 7.5% convertible monthly to make up the shortfall not covered by the first loan. Find the amount of the monthly payment on the second loan.

Example (Exercise 3.4)

A borrows \$20,000 for 8 years and repays the loan with level annual payments at the end of each year. B also borrows \$20,000 for 8 years. but pays only interest as it is due each year and plans to repay the entire loan at the end of the 8-year period. Both loans carry an effective interest rate of 8.5%. How much more interest will B pay than A pays over the life of the loan?

Annuity-Due I

Definition

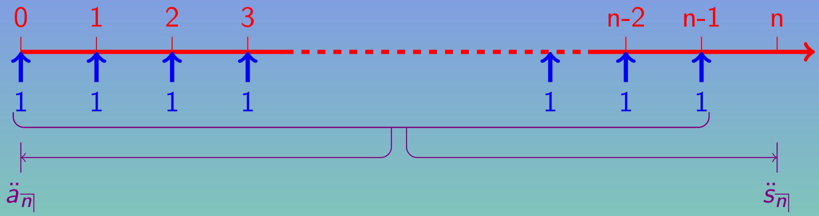
Annuity-Due: Payments are made at the beginning of each of the n periods.
The present value at $t = 0$ for payments 1's is denoted by

$$\ddot{a}_{\overline{n}|}$$

The accumulated value at $t = n$ (one period after the last payment is made) for payments 1's is denoted by

$$\ddot{s}_{\overline{n}|}$$

Annuity-Due II



So

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d}$$

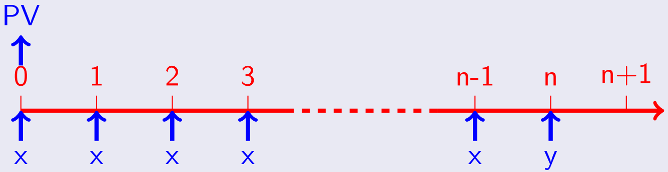
$$\ddot{s}_{\overline{n}|} = u\frac{u^n - 1}{u - 1} = \frac{u^n - 1}{1 - \frac{1}{u}} = \frac{u^n - 1}{d}.$$

Annuity-Due III

Use BEGINNING mode in a financial calculator to:

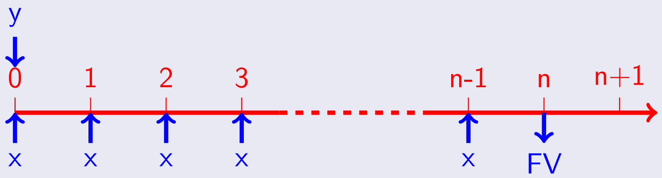
calculate $PV = x\ddot{a}_{\overline{n}|} + yv^n$, or $PV - x\ddot{a}_{\overline{n}|} - yv^n = 0$,

$$\text{input: } \; PMT = -x, \; FV = -y, \; \text{ output: } \; PV.$$



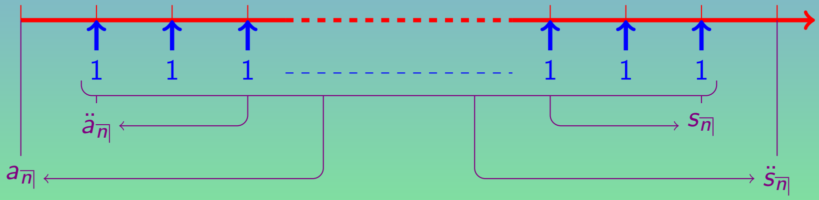
calculate $FV = x\ddot{s}_{\overline{n}|} + yu^n$, or $FV - x\ddot{s}_{\overline{n}|} - yu^n = 0$,

$$\text{input:}\quad PMT = -x,\ PV = -y,\quad \text{output:}\quad FV.$$



Annuity-Due V

The positions of annuity-immediate and annuity-due relative to the payments are the following:



Example (Exercise 3.7)

Find $\ddot{a}_{\overline{n}|}$ if the effective rate of discount is 10%.

Example (Exercise 3.8)

Find the present value of payments of \$200 every six months starting immediately and continuing through four years from the present, and \$100 every six months thereafter through ten years from the present, if $i^{(2)} = 0.06$.

Example (Exercise 3.9)

A worker aged 40 wishes to accumulate a fund for retirement by depositing $\$3,000$ at the beginning of each year for 25 years. Starting at age 65 the worker plans to make 15 annual withdrawals at the beginning of each year. Assuming that all payments are certain to be made, find the amount of each withdrawal starting at age 65, if the effective rate of interest is 8% during the first 25 years but only 7% thereafter.

Annuity Values on Any Date I

Example (Exercise 3.15)

Annuities X and Y provide the following payments:

End of Year	Annuity X	Annuity Y
$1-10$	1	K
$11-20$	2	0
$21-30$	1	K

Annuities X and Y have equal present values at an annual effective interest rate i such that $v^{10}=1/2$. Determine K.

Example (Exercise 3.17)

Find the present value on January 1 of an annuity which pays \$2000 every six months for five years. The first payment is due on the next April 1 and the rate of interest is 9% convertible semiannually.

Perpetuities I

Definition

A perpetuity is an annuity with infinitely many payments.

Let $n \to \infty$ in $1 + v + v^2 + \cdots + v^{n-1} = \frac{1-v^n}{1-v}$ (note that $|v| < 1$, so $v^n \to 0$ as $n \to \infty$) we have

The value of a perpetuity of payment 1 at the time of the first payment is

$$1 + v + v^2 + \cdots = \frac{1}{1-v}.$$

Perpetuities II

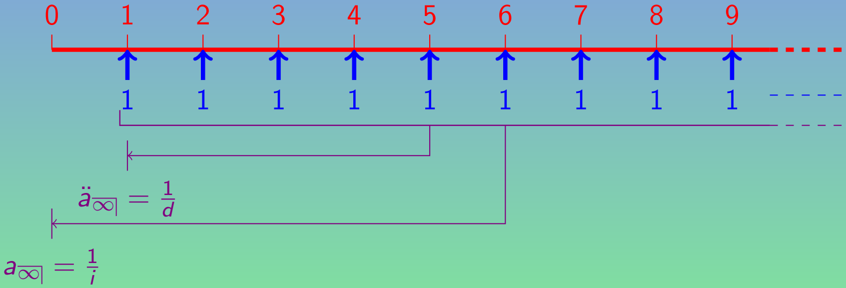
So we have

$$a_{\overline{\infty}|} = \frac{v}{1-v} = \frac{1}{i}$$

and

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{1-v} = \frac{1}{d}.$$

Perpetuities III



Example (Exercise 3.18)

Deposits of \$1000 are placed into a fund at the beginning of each year for the next 20 years. After 30 years annual payments commence and continue forever, with the first payment at the end of the 30th year. Find an expression for the amount of each payment.

Example (Exercise 3.19)

A deferred perpetuity-due begins payments at time n with annual payments of \$1000 per year. If the present value of this perpetuity-due is equal to \$6561 and the effective rate of interest $i = 1/9$, find n.

Example (Exercise 3.20)

A woman has an inheritance in a trust fund for family members left by her recently deceased father that will pay $50,000 at the end of each year indefinitely into the future. She has just turned 60 and does not think that this perpetuity-immediate meets her retirement needs. She wishes to exchange the value of her inheritance in the trust fund for one which will pay her a 5-year deferred annuity-immediate providing her a retirement annuity with annual payments at the end of each year for 20 years following the 5-year deferral period. She would have no remaining interest in the trust fund after 20 payments are made. If the trustee agrees to her proposal, how much annual retirement income would she receive? The trust fund is earning an annual effective rate of interest equal to 5%. Answer to the nearest dollar.

Unknown Time I

Question:

For a loan with fixed monthly payments, how long will it take to pay off the loan?

We can easily find out n by using the formula we learned. However the answer is most likely not an integer. So after making n regular payments, you may owe an amount smaller than the regular payment. In practice, there are two ways to treat such a smaller payment: either combine it with the last payment (balloon payment), or make a smaller payment at the next scheduled time (drop payment).

Unknown Time II

Since the FV in the BEGINNING mode of a TVM solver is the balance one period after the last regular payment, we can easily calculate both balloon and drop payments using the BEGINNING mode.

Calculating Balloon or Drop Payment Using TVM Solver

- Calculate n
- Use BEGINNING mode:

$$N = \lfloor n \rfloor - 1, \quad \text{then} \quad FV = \text{balloon payment.}$$

$$N = \lfloor n \rfloor, \quad \text{then} \quad FV = \text{drop payment.}$$

where $\lfloor n \rfloor$ is the largest integer $\leq n$.

Example (Exercise 3.24)

A loan of \$1000 is to be repaid by annual payments of \$100 to commence at the end of the fifth year and to continue thereafter for as long as necessary. Find the time and amount of the final payment, if the final payment is to be larger than the regular payments. Assume $i = 4.5\%$.

Example (Exercise 3.26)

A fund earning 8% effective is being accumulated with payments of \$500 at the beginning of each year for 20 years. Find the maximum number of withdrawals of \$1000 that can be made at the ends of years under the condition that once withdrawals start they must continue through the end of the 20-year period.

Unknown Rate of Interest I

We can use a TVM solver to calculate the interest i easily once we know PV, PMT, FV, and n.

There is also an approximated formula by using a linearization of $a_{\overline{n}|}$.

For a given present value PV and payments PMTs, we have an equation,

$$(PMT)a_{\overline{n}|} = PV, \quad \text{or} \quad a_{\overline{n}|} = \frac{PV}{PMT} = g.$$

Let i be the interest rate per PAYMENT PERIOD. Then

$$i \approx \frac{2(n-g)}{g(n+1)}. \tag{3.21}$$

Unknown Rate of Interest II

Consider $a_{\overline{n}|} = g$. We have

$$\frac{1 - \frac{1}{(1+i)^n}}{i} = \frac{(1+i)^n - 1}{i(1+i)^n} = g,$$

or

$$\frac{(1+i)^n}{\frac{(1+i)^n - 1}{i}} = \frac{1}{g}.$$

Let

$$h(i) = \frac{(1+i)^n - 1}{i}$$

and

$$f(i) = \frac{(1+i)^n}{h(i)}.$$

Unknown Rate of Interest III

Note that

$$
\begin{aligned}
h(i) &= \frac{(1+i)^n - 1}{i} \\
&= \frac{(1 + ni + \frac{n(n-1)}{2} i^2 + \frac{n(n-1)(n-2)}{3!} i^3 \cdots + i^n) - 1}{i} \\
&= n + \frac{n(n-1)}{2} i + \frac{n(n-1)(n-2)}{3!} i^2 + \cdots + i^{n-1}
\end{aligned}
$$

So

$$h(0) = n, \quad h'(0) = \frac{n(n-1)}{2},$$

$$f(0) = \frac{(1+0)^n}{h(0)} = \frac{1}{n}$$

Unknown Rate of Interest IV

$$
\begin{aligned}
f'(0) &= \frac{h(0)n(1+0)^{n-1} - (1+0)^n h'(0)}{h^2(0)} \\
&= \frac{n^2 - \frac{n(n-1)}{2}}{n^2} = 1 - \frac{n-1}{2n} = \frac{n+1}{2n}
\end{aligned}
$$

Unknown Rate of Interest V

We have

$$\frac{1}{g} = f(i) \approx f(0) + f'(0)i = \frac{1}{n} + \frac{n+1}{2n}i$$

and from it we can solve

$$i \approx \frac{2(n-g)}{g(n+1)}.$$

Example (Exercise 3.28)

A 48-month car loan of $\$12,000$ can be completely paid off with monthly payments of $\$300$ made at the end of each month. What is the nominal rate of interest convertible monthly on this loan?

a) Computed on an exact basis with a financial calculator.

b) Approximated by formula

$$i \approx \frac{2(n-g)}{g(n+1)}.$$

Example (Exercise 3.30)

A beneficiary receives a $\$10,000$ life insurance benefit. If the beneficiary uses the proceeds to buy a 10-year annuity-immediate, the annual payout will be $\$1538$. If a 20-year annuity-immediate is purchased, the annual payout will be $\$1072$. Both calculations are based on an annual effective interest rate of i. Find i.

Varying Interest

Definition

Portfolio Rate Method: (Interest rate is attached to period)
A particular interest rate only be applied to a particular period, and any payment discount or accumulate through that period will be applied to the corresponding interest rate.

Definition

Yield Curve Method: (Interest rate is attached to payment)
A particular interest rate only be applied to a particular payment, and the same interest rate will be applied to that particular payment through any period.

Example (Exercise 3.32)

a) Find the present value of an annuity-immediate which pays 1 at the end of each half-year for five years, if the rate of interest is 8% convertible semiannually for the first three years and 7% convertible semiannually for the last two years. (I.e. Portfolio method.)

b) Find the present value of an annuity-immediate which pays 1 at the end of each half-year for five years, if the payments for the first three years are discounted at the interest rate 8% convertible semiannually and the payments for the last two years are discounted at the interest rate 7% convertible semiannually. (I.e. Yield curve method.)

Example (Exercise 3.34)

A loan of P is to be repaid by 10 annual payments beginning 6 months from the date of the loan. The first payment is to be half as large as the others. For the first $4\frac{1}{2}$ years interest is at i effective; for the remainder of the term interest is at j effective. Find an expression for the first payment.