## Macaulay and Modified Durations I

For a given cash flow $\left\{\left(a_{k}, t_{k}\right): k \in \mathbb{N}\right\}$, its present value

$$
p(i)=\sum_{k} \frac{a_{k}}{(1+i)^{t_{k}}}
$$

is a function of the effective interest rate $i$.

## Macaulay and Modified Durations II

## Definition

For a given cash flow $\left\{\left(a_{k}, t_{k}\right): k \in \mathbb{N}\right\}$, the Macaulay duration is defined by

$$
d_{m a c}=\frac{\sum_{k} t_{k} \frac{a_{k}}{(1+i)^{t_{k}}}}{\sum_{k} \frac{a_{k}}{(1+i)^{t_{k}}}}=\frac{\sum_{k} t_{k} \frac{a_{k}}{(1+i)^{t_{k}}}}{p(i)}
$$

and the modified duration is defined by

$$
d_{\text {mod }}=\frac{-p^{\prime}(i)}{p(i)}=\frac{\sum_{k} t_{k} \frac{a_{k}}{(1+i)^{t_{k}+1}}}{p(i)}
$$

Clearly

$$
d_{m o d}=\frac{d_{m a c}}{1+i}
$$

## Macaulay and Modified Durations III

From the definition we can see that if there are two cash flows $A=\left\{\left(t_{k}, a_{k}\right)\right\}$ and $B=\left\{\left(t_{k}, b_{k}\right)\right\}$, then the Macaulay duration of the cash flow $A+B$ satisfies

$$
\begin{aligned}
d_{\text {mac }, A+B} & =\frac{\sum_{k} t_{k} \frac{a_{k}+b_{k}}{(1+i)^{t_{k}}}}{p_{A}(i)+p_{B}(i)} \\
& =\frac{p_{A}(i) \frac{\sum_{k} t_{k} \frac{a_{k}}{(1+i)^{t_{k}}}}{p_{A}(i)}+p_{B}(i) \frac{\sum_{k} t_{k} \frac{b_{k}}{(1+i)^{t_{k}}}}{p_{B}(i)}}{p_{A}(i)+p_{B}(i)} \\
& =\frac{p_{A}(i) d_{\text {mac }, A}+p_{B}(i) d_{\text {mac }, B}}{p_{A}(i)+p_{B}(i)}
\end{aligned}
$$

## Macaulay and Modified Durations IV

So we have

$$
d_{\operatorname{mac}, A+B}=\frac{p_{A}(i) d_{\operatorname{mac}, A}+p_{B}(i) d_{\operatorname{mac}, B}}{p_{A}(i)+p_{B}(i)}
$$

## Macaulay and Modified Durations V

## Definition

For a given cash flow $\left\{\left(a_{k}, t_{k}\right): k \in \mathbb{N}\right\}$, the Macaulay convexity is defined by

$$
c_{\text {mac }}(i)=\frac{\sum_{k} t_{k}^{2} \frac{a_{k}}{(1+i)^{t_{k}}}}{p(i)} .
$$

and the modified convexity is defined by

$$
c_{\text {mod }}(i)=\frac{p^{\prime \prime}(i)}{p(i)}=\frac{\sum_{k} t_{k}\left(t_{k}+1\right) \frac{a_{k}}{(1+i)^{t_{k}+2}}}{p(i)}
$$

## Macaulay and Modified Durations VI

Example (Exam FM Sample Question 122)
Cash flows are 40,000 at time 2 (in years), 25,000 at time 3, and 100,000 at time 4. The annual effective yield rate is $7.0 \%$.
Calculate the Macaulay duration.

## Macaulay and Modified Durations VII

## Example (Exam FM Sample Question 121)

Annuity A pays 1 at the beginning of each year for three years. Annuity $B$ pays 1 at the beginning of each year for four years. The Macaulay duration of Annuity $A$ at the time of purchase is 0.93 . Both annuities offer the same yield rate. Calculate the Macaulay duration of Annuity B at the time of purchase.

## Macaulay and Modified Durations VIII

## Example (Exam FM Sample Question 124)

Rhonda purchases a perpetuity providing a payment of 1 at the beginning of each year. The perpetuity's Macaulay duration is 30 years.
Calculate the modified duration of this perpetuity.

## First Order Macaulay and Modified Approximations I

By the Taylor's formula,

$$
\begin{aligned}
p(i) & \approx p\left(i_{0}\right)+p^{\prime}\left(i_{0}\right)\left(i-i_{0}\right) \\
& =p\left(i_{0}\right)-p\left(i_{0}\right) d_{\text {mod }}\left(i_{0}\right)\left(i-i_{0}\right) \\
& =p\left(i_{0}\right)\left(1-d_{\text {mod }}\left(i_{0}\right)\left(i-i_{0}\right)\right) .
\end{aligned}
$$

So the first order modified approximation of $p(i)$ around $i_{0}$ is given by

$$
p(i) \approx p\left(i_{0}\right)\left(1-d_{\text {mod }}\left(i_{0}\right)\left(i-i_{0}\right)\right) .
$$

## First Order Macaulay and Modified Approximations II

To find the first order Macaulay approximation, consider the current value $c(i, t)$ of cash flow at $t$,

$$
c(i, t)=(1+i)^{t} p(i)=\sum_{k} a_{k}(1+i)^{t-t_{k}},
$$

and assume $a_{k}>0$ for all $k$.
Since

$$
\frac{\partial c}{\partial i}(i, 0)=\sum_{k} a_{k}\left(-t_{k}\right)(1+i)^{-t_{k}-1}<0
$$

and

$$
\lim _{t \rightarrow \infty} \frac{\partial c}{\partial i}(i, t)=\lim _{t \rightarrow \infty} \sum_{k} a_{k}\left(t-t_{k}\right)(1+i)^{t-t_{k}-1}>0
$$

## First Order Macaulay and Modified Approximations III

there exists a $T>0$ such that $\frac{\partial c}{\partial i}(i, T)=0$. Let

$$
\frac{\partial c}{\partial i}(i, T)=T(1+i)^{T-1} p(i)+(1+i)^{T} p^{\prime}(i)=0
$$

Then

$$
T(i)=-(1+i) \frac{p^{\prime}(i)}{p(i)}=(1+i) d_{\bmod }(i)=d_{\operatorname{mac}}(i)
$$

Now, by the Taylor's formula

$$
c\left(i, d_{\operatorname{mac}}\left(i_{0}\right)\right) \approx c\left(i_{0}\right)+\frac{\partial c}{\partial i}\left(i_{0}, d_{\operatorname{mac}}\left(i_{0}\right)\right)\left(t-t_{0}\right)=c\left(i_{0}\right)
$$

i.e.,

$$
(1+i)^{d_{\max }} p(i) \approx\left(1+i_{0}\right)^{t} p\left(i_{0}\right)
$$

## First Order Macaulay and Modified Approximations IV

So we have the first order Macaulay approximation:

$$
p(i) \approx p\left(t_{0}\right)\left(\frac{1+i_{0}}{1+i}\right)^{d_{m a c}}
$$

## First Order Macaulay and Modified Approximations V

## Example (Exam FM Sample Question 178)

A 20-year bond priced to have an annual effective yield of $10 \%$ has a Macaulay duration of 11 . Immediately after the bond is priced, the market yield rate increases by $0.25 \%$. Calculate the bond's approximate percentage price change using a first-order Macaulay approximation.

## First Order Macaulay and Modified Approximations VI

## Example (Exam FM Sample Question 189)

A bond has a modified duration of 8 and a price of 112,955 calculated using an annual effective interest rate of $6.4 \%$. $E_{M A C}$ is the estimated price of this bond at an interest rate of $7.0 \%$ using the first-order Macaulay approximation. $E_{M O D}$ is the estimated price of this bond at an interest rate of $7.0 \%$ using the first-order modified approximation. Calculate $E_{M A C}-E_{M O D}$.

## First Order Macaulay and Modified Approximations VII

## Example (Exam FM Sample Question 190)

SOA Life Insurance Life Insurance Company has a portfolio of two bonds:

- Bond 1 is a bond with a Macaulay duration of 7.28 and a price of 35,000 ; and
- Bond 2 is a bond with a Macaulay duration of 12.74 and a price of 65,000.
The price and Macaulay duration for both bonds were calculated using an annual effective interest rate of $4.32 \%$. Bailey estimates the value of this portfolio at an interest rate of $i$ using the first-order Macaulay approximation to be 105,000. Determine $i$.


## Cash-flow Matching I

The Cash-flow Matching is to structure an asset portfolio in such a fashion that the cash inflow that will be generated will exactly match the cash outflow from the liabilities in every period.

## Cash-flow Matching II

## Example (Exam FM Sample Question 181)

A company has liabilities that require it to make payments of 1000 at the end of each of the next five years. The only investments available to the company are as follows:

| Investment | Price | Subsequent Cash Flows |
| :---: | :---: | :---: |
| J | 1500 | 500 at the end of each year for 5 years |
| K | 500 | 1000 at the end of year 5 |
| L | 1000 | 500 at the end of each year for 4 years |
| M | 4000 | 1000 at the end of each year for 5 years |

The company is able to purchase as many of each investment as it wants, but only in whole units. The company's investment objective is to be fully immunized over the next five years.
Calculate the lowest possible cost to achieve this objective.

## Cash-flow Matching III

## Example (Exam FM Sample Question 132)

A bank accepts a 20,000 deposit from a customer on which it guarantees to pay an annual effective interest rate of $10 \%$ for two years. The customer needs to withdraw half of the accumulated value at the end of the first year. The customer will withdraw the remaining value at the end of the second year. The bank has the following investment options available, which may be purchased in any quantity:

Bond H: A one-year zero-coupon bond yielding 10\% annually
Bond I: A two-year zero-coupon bond yielding 11\% annually
Bond J: A two-year bond that sells at par with $12 \%$ annual coupons
Any portion of the 20,000 deposit that is not needed to be invested in bonds is retained by the bank as profit. Determine the investment strategies produces the highest profit for the bank, and is guaranteed to meet the customer's withdrawal needs.

## Redington Immunization I

Redington immunization assumes the yield curve is flat and the changes are parallel, i.e., the interests of all zero coupon bonds are the same and change the same amount regardless their durations. The first assumption for the Redington immunization is that the present values of cash inflow and outflow are equal, i.e.

$$
p(i)=0
$$

To achieve the Redington immunization, $p(i)=0$ is a local minimum, i.e.,

$$
p(i+\epsilon) \geq 0, \quad \text { for small }|\epsilon|
$$

(the present value of the cash inflow is always greater than or equal to present value of the cash outflow no matter which directions of the interest rate moves).

## Redington Immunization II

From the calculus, we know that the conditions are:

- $p^{\prime}(i)=0$,
- $p^{\prime \prime}(i)>0$.


## Redington Immunization III

So the Redington immunization strategy is to arrange the assets so that:

1) The present value of the cash inflows from the assets equals the present value of the cash outflows from the liabilities.
2) The duration of the assets equals the duration of the liabilities (either Macaulay or modified, and only the numerators of the durations).
3) The convexity (either Macaulay or modified, and only the numerators of the convexity) of the assets is greater than the convexity of the liabilities.

## Redington Immunization IV

## Redington Immunization

Let $\left\{\left(L_{j}, t_{j}\right): j=1, \ldots, n\right\}$ be the liabilities. The Redington immunization strategy is to arrange the assets providing cash inflows $\left\{\left(A_{k}, s_{k}\right): k=1, \ldots, m\right\}$ so that:

1) $A_{1} v^{s_{1}}+\cdots+A_{m} v^{s_{m}}=L_{1} v^{t_{1}}+\cdots+L_{n} v^{t_{n}}$,
2) $s_{1} A_{1} v^{s_{1}}+\cdots+s_{m} A_{m} v^{s_{m}}=t_{1} L_{1} v^{t_{1}}+\cdots+t_{n} L_{n} v^{t_{n}}$, or the durations (either Macaulay or modified) of the assets and liabilities are equal.
3) $s_{1}^{2} A_{1} v^{s_{1}}+\cdots+s_{m}^{2} A_{m} v^{s_{m}}>t_{1}^{2} L_{1} v^{t_{1}}+\cdots+t_{n}^{2} L_{n} v^{t_{n}}$, or the convexity of the assets is greater than the convexity of the liabilities,
where $v=\frac{1}{1+i_{0}}$ is the current discount factor.

## Redington Immunization V

## Example (Exam FM Sample Question 73)

Trevor has assets at time 2 of $A$ and at time 9 of $B$. He has a liability of 95,000 at time 5. Trevor has achieved Redington immunization in his portfolio using an annual effective interest rate of $4 \%$. Calculate $\frac{A}{B}$.

## Redington Immunization VI

## Example (Exam FM Sample Question 127)

A company owes 500 and 1000 to be paid at the end of year one and year four, respectively. The company will set up an investment program to match the duration and the present value of the above obligation using an annual effective interest rate of $10 \%$. The investment program produces asset cash flows of $X$ today and $Y$ in three years. Calculate $X$ and determine whether the investment program satisfies the conditions for Redington immunization.

## Redington Immunization VII

## Example (Exam FM Sample Question 182)

A railroad company is required to pay 79,860 , which is due three years from now. The company invests 15,000 in a bond with modified duration 1.80, and 45,000 in a bond with modified duration $d_{\text {mod }}$, to Redington immunize its position against small changes in the yield rate. The annual effective yield rate for each of the bonds is $10 \%$. Calculate $d_{\text {mod }}$.

## Full Immunization I

Redington immunization technique is designed to work for small changes in $i$. Full immunization can be applied for changes of any magnitude in $i$.
Consider a liability cash outflow of $L$ at the time $t$. The concept of full immunization is to hold assets providing cash inflows of $A$ and $B$ at times $t-a$ and $t+b$, respectively, where $a, b>0$, such that the current value function at $t$

$$
p(i)=A(1+i)^{a}+B(1+i)^{-b}-L
$$

satisfies two conditions:

## Full Immunization II

1) $p\left(i_{0}\right)=0$, i.e.,

$$
A\left(1+i_{0}\right)^{a}+B\left(1+i_{0}\right)^{-b}=L
$$

2) $p^{\prime}\left(i_{0}\right)=0$, i.e.,

$$
a A\left(1+i_{0}\right)^{a-1}-b B\left(1+i_{0}\right)^{-b-1}=0
$$

or

$$
a A\left(1+i_{0}\right)^{a}-b B\left(1+i_{0}\right)^{-b}=0
$$

## Full Immunization III

## Full Immunization

Let $L$ be a liability at the time $t$. The full immunization strategy is to arrange the assets providing cash inflows $A$ and $B$ at the times $t-a$ and $t+b, a, b>0$, respectively, so that:

1) $A u^{a}+B u^{-b}=L$,
2) $a A u^{a}-b B u^{-b}=0$, where $u=1+i_{0}$ is the current accumulation factor.

## Full Immunization IV

To see that the full immunization achieves the desired result, let's re-write $p(i)$ as

$$
\begin{aligned}
p(i) & =A(1+i)^{a}+B(1+i)^{-b}-L \\
& =A(1+i)^{a}+B(1+i)^{-b}-\left(A\left(1+i_{0}\right)^{a}+B\left(1+i_{0}\right)^{-b}\right)
\end{aligned}
$$

Since $B\left(1+i_{0}\right)^{-b}=\frac{a}{b} A\left(1+i_{0}\right)^{a}$,

$$
\begin{aligned}
p(i)= & A(1+i)^{a}+B(1+i)^{-b}-\left(A\left(1+i_{0}\right)^{a}+B\left(1+i_{0}\right)^{-b}\right) \\
= & A(1+i)^{a}+\frac{a}{b} A\left(1+i_{0}\right)^{a}\left(\frac{1+i}{1+i_{0}}\right)^{-b} \\
& \quad-\left(A\left(1+i_{0}\right)^{a}+\frac{a}{b} A\left(1+i_{0}\right)^{a}\right) \\
= & A\left(1+i_{0}\right)^{a}\left(\left(\frac{1+i}{1+i_{0}}\right)^{a}+\frac{a}{b}\left(\frac{1+i}{1+i_{0}}\right)^{-b}-\left(1+\frac{a}{b}\right)\right)
\end{aligned}
$$

## Full Immunization V

Consider the function

$$
f(x)=x^{a}+\frac{a}{b} x^{-b}-\left(1+\frac{a}{b}\right)
$$

We have

$$
\begin{gathered}
f(1)=0 \\
f^{\prime}(x)=a\left(x^{a-1}-x^{-b-1}\right)=\frac{a}{x}\left(x^{a}-x^{-b}\right)
\end{gathered}
$$

So $f^{\prime}(x)<0$ for all $x \in(0,1), f^{\prime}(x)>0$ for all $x \in(1, \infty)$, and therefore $f(1)=0$ is the global minimum value for all $x \in(0, \infty)$.

## Full Immunization VI

Since

$$
p(i)=A\left(1+i_{0}\right)^{a} f\left(\frac{1+i}{1+i_{0}}\right)
$$

we have

$$
p\left(i_{0}\right)=0 \text { and } p(i)>0 \text { for all } i \neq i_{0} .
$$

## Full Immunization VII

## Example (Exam FM Sample Question 71)

Aakash has a liability of 6000 due in four years. This liability will be met with payments of $A$ in two years and $B$ in six years. Aakash is employing a full immunization strategy using an annual effective interest rate of $5 \%$. Calculate $|A-B|$.

## Full Immunization VIII

## Example (Exam FM Sample Question 72)

Jia Wen has a liability of 12,000 due in eight years. This liability will be met with payments of 5000 in five years and $B$ in $8+b$ years. Jia Wen is employing a full immunization strategy using an annual effective interest rate of $3 \%$. Calculate $\frac{B}{b}$.

