EARTHQUAKES AND CIRCLE PACKINGS

G. BROCK WILLIAMS

ABSTRACT. We prove that earthquakes on hyperbolic surfaces can be approximated by discrete earthquakes constructed using circle packings. Consequently, we find a combinatorial version of Thurston's Earthquake Theorem. Any surface can be approximated by combinatorial earthquakes of a packable surface. This provides a controlled combinatorial method of deforming hyperbolic surfaces. We also find a stronger version of the density results of Brooks and Bowers and Stephenson by showing that combinatorial earthquakes of any single packable surface generate a dense subset of Teichmüller space.

1. INTRODUCTION

A circle packing is a configuration of circles with a prescribed pattern of tangencies. The two parts of this definition point to the tension between the dual natures of circle packings. "A configuration of circles" suggests an inherently geometric structure; indeed, the definition of a circle itself presupposes the existence of a metric. Our configurations will be deeply connected to the geometry of the underlying surface, at times being constrained by, and at other times creating, the geometry in which they live.

The "prescribed pattern" is a purely combinatorial notion. Usually described by a graph or simplicial complex, this abstract pattern is taken to be the starting point in a packing problem. As the circles struggle to fit together, they translate these combinatorial constraints into concrete geometric structures.

The deep connection to geometric function theory first appeared in Thurston's study [34] of packings with the same combinatorial structure, but different radii, and hence, different geometric structures. The shared combinatorial structure permits the construction of maps between the different packings. Amazingly, such maps are approximately conformal. The circles somehow impose *conformally invariant* geometry. Refinements of Thurston's original observation have been made by several authors [18, 31], with the most general version due to He and Schramm [19].

Several recent investigations [5, 8, 37, 38] have focused on the effect of combinatorial subdivision or other deformations on the geometry of the resulting packing. We will consider herein the effect of combinatorial deformations which act like earthquakes.

The notion of an earthquake was developed by Thurston [33, 35] and used by Kerckhoff in his proof of the Nielsen Realization Conjecture [21] and by Bonahon [4] and Papadopoulos [28] in their study of Thurston's boundary of Teichüller space. Briefly, an earthquake cuts a surface along a measured geodesic lamination, shears the surface to the left, and then re-attaches the surface. Thurston showed that

Date: November 15, 2000.

1991 *Mathematics Subject Classification.* 52C26,30F60.

Key words and phrases. Circle Packing, Earthquakes, Teichmüller Space.

every homeomorphism of the boundary of the unit disk onto itself can be realized as the boundary values of an earthquake of the disk. As a result, any two complete hyperbolic surfaces (of the same topological type) can be deformed into one another by an earthquake [33, 35].

After introducing some of the relevant background material in Sections 2 and 3, we develop a general form of combinatorial attachment in Section 4. This operation permits us to cut open an abstract complex, shear it to the left, and re-attach the pieces. The abstract complex which results from such a *combinatorial earthquake*, will have no geometric structure on its own, but when realized by a circle packing, the new geometry imposed by the circles will approximate the effect of a classical earthquake. In particular, we will consider earthquakes of the unit disk $\mathbb{D}$, and in Section 5 will prove the following:

Theorem 1. Given any finite measured geodesic lamination $\mathcal{L}$ and a sequence of finite bounded degree packings P_n in $\mathbb{D}$ with mesh decreasing to 0, the discrete earthquake maps E_n formed by combinatorial earthquakes of P_n converge locally uniformly to the earthquake map E induced by $\mathcal{L}$.

Circle packings also exist for Riemann surfaces; indeed, for any reasonable complex, there is a unique Riemann surface which supports a packing with the prescribed pattern [3]. Moreover, Brooks [9] and Bowers and Stephenson [6, 7] have shown that the packable surfaces are dense in Teichmüller space. However, as many authors have noted [3, 6, 7, 10, 30], moving beyond the fact of their density to determining exactly how the combinatorics affects the geometry has proven to be a very difficult question.

In Section 6, we examine the impact of a combinatorial earthquake on the conformal structure of the surface which supports the packing. We prove a combinatorial version of Thurston's Earthquake Theorem [35] and see that we can shake our way through Teichmüller space using combinatorial earthquakes.

Theorem 2. Let $\mathcal{K}_0$ be a marked abstract triangulation of a surface of topological type (g, n). Then $\mathcal{K}_0$ generates a dense packable subset of Teichmüller space by hex refinement and combinatorial earthquakes. That is, for any Riemann surface R of type (g, n) there is a sequence of combinatorial earthquakes and refinements of $\mathcal{K}_0$ which produce packable surfaces converging to R.

Thus, not only are the packable surfaces dense [6, 7, 9], but any one packable surface can be used to create a dense family using only earthquakes and hex refinements.

Finally, we would like to thank Fred Gardiner and Nikola Lakic for very helpful conversations during the preparation of this manuscript.

2. Earthquakes

2.1. **Riemann Surfaces.**

We begin by reviewing some background material on Riemann surfaces and their deformation spaces. Some of the many excellent references include [11, 16, 17, 20, 24, 27].

Definition 2.1. A **Riemann surface** is a complex one-dimensional manifold with charts whose overlap maps are conformal.

A first attempt at classifying Riemann surfaces might be by their topological type. If a surface R has precisely g handles, n punctures, and m infinite-area ends, then R is of said to be of type (g, n, m). (If m is 0, then it is often omitted.) The type determines a surface's universal cover and the sign of the curvature of the complete metric it supports. For example, a surface of type (g, n) with $2g-2+n > 0$, has the hyperbolic plane as its universal cover and supports a complete metric of constant negative curvature. Such a surface will be referred to as a hyperbolic surface.

Unfortunately, such topological characteristics do not tell a surface's entire story. We need an equivalence relation that keeps track of more information. Equivalent surfaces should share the same conformal structure and generators of their fundamental groups should also correspond. Such a choice of generators is called a *marking*.

Alternatively, the equivalence relation can be defined using quasiconformal maps of a fixed reference surface. We define quasiconformal maps f_1 and f_2 of a Riemann surface R to be equivalent if $f_2 \circ (f_1)^{-1}$ is homotopic (modulo the boundary) to a conformal map.

Definition 2.2. Fix a reference Riemann surface R and suppose f_1 and f_2 are quasiconformal maps from R to R_1 and R_2, respectively. We say R_1 and R_2 are equivalent if f_1 and f_2 are equivalent mappings. The **Teichmüller space** of R is set of equivalence classes.

The equivalence relation is independent of the reference surface R — any other Riemann surface of the same topological type would generate the same Teichmüller space.

There is a natural metric on Teichmüller space determined by how nearly conformal homeomorphisms between surfaces may be. More specifically, the distance between surfaces $R_1 = f_1(R)$ and $R_2 = f_2(R)$ is given by

$$\frac{1}{2}\log K^*,$$

where K^* is the infimum of the dilatation of $g_2 \circ (g_1)^{-1}$ with g_i equivalent to f_i, $i = 1, 2$.

Teichmuller's Theorem. Between any two points R_1 and R_2 in Teichmüller space there is a unique quasiconformal map T_{R_1,R_2} of minimal dilatation. This map is called the **Teichmüller map** from R_1 to R_2.

Notice that R_1 and R_2 are equivalent iff T_{R_1,R_2} is homotopic to a conformal map.

2.2. **Laminations and Earthquake Maps.** A hyperbolic Möbius transformation of the hyperbolic plane (modeled as the unit disk $\mathbb{D}$) fixes two points of $\partial\mathbb{D}$ and translates every point on the geodesic joining the fixed points by some fixed hyperbolic distance. Thus a geodesic L and a nonnegative weight σ determine a shearing transformation of $\mathbb{D}$ in the following manner: Cut $\mathbb{D}$ along L. Apply to one of the resulting half planes the hyperbolic Möbius transformation which translates along L a distance σ. Now re-attach the half-planes. Since L is invariant under the shearing Möbius transformation, the two halves will still fit together to re-create $\mathbb{D}$. There is of course, some ambiguity as to which direction to shear before re-attaching the half-planes. We will always shear in such a way that each half-plane appears to have moved to the left relative to the other. Notice that the idea of "left" requires only an orientation on $\mathbb{D}$, not on L. See Figure 1.

This shearing operation can easily be extended to finite collections of disjoint weighted geodesics and, by lifting to the universal cover, to geodesics on hyperbolic surfaces. This type of deformation dates to Fenchel and Nielsen's description of

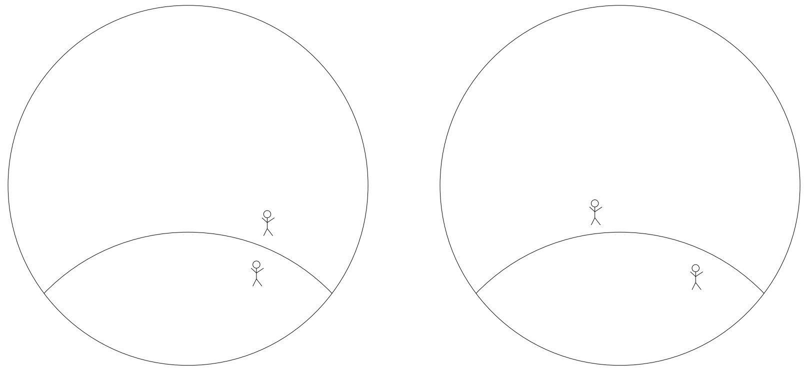

FIGURE 1. The effect of shearing to the left along a geodesic. Notice that it appears to each observer that the other has moved left.

coordinates for Teichmüller space [11, 20]. The resulting transformation of the surface is called a **simple left earthquake** [17, 35]. The most important earthquakes, however, do not occur along finite collections of geodesics, but along more general measured geodesic laminations [12, 15, 21, 33, 35, 36].

Definition 2.3. A **geodesic lamination** $\mathcal{L}$ of a Riemann surface R is a closed subset of R consisting of a collection of geodesics foliated in the sense that every point of $\mathcal{L}$ has a neighborhood U and a continuous map $\phi_i : U \cap \mathcal{L} \to (0,1) \times B$ such that the overlap maps are of the form $\phi_i \phi_j^{-1} = (f(x,y), g(y))$.

If B is a discrete set, then $\mathcal{L}$ is a **discrete lamination**. The geodesics comprising $\mathcal{L}$ are called **leaves**. The components of the complement of the leaves are the **gaps**; the gaps together with the leaves that bound them form the **strata**.

Definition 2.4. A **measured geodesic lamination** additionally possesses a transverse measure invariant under homotopies preserving the geodesics of $\mathcal{L}$.

The notion of a earthquake along a measured geodesic lamination was developed by Thurston. His original definition was first published by Kerckhoff [21]; the definition below is taken from Thurston's later work [35].

Definition 2.5. If $\mathcal{L}$ is a measured geodesic lamination of $\mathbb{D}$, the left **earthquake** E induced by $\mathcal{L}$ is the homeomorphism of $\mathbb{D}$ onto itself which satisfies

(1) E restricted to any stratum of $\mathcal{L}$ is a hyperbolic isometry
(2) On any geodesic $L^{i,j}$ separating stratum S^i on its left from stratum S^j on its right, the comparison isometry

$$H^{i,j} = (E|_{S^i})^{-1} \circ E|_{S^j} \tag{2.1}$$

is the hyperbolic Möbius transformation with axis $L^{i,j}$ and translation distance determined by the measure on $L^{i,j}$.

Now by lifting to the universal cover, we can consider earthquakes on hyperbolic surfaces.

Definition 2.6. If R is a complete hyperbolic surface with measured geodesic lamination $\mathcal{L}$, an **earthquake** on R is a homeomorphism to another hyperbolic surface of the same type which lifts to an earthquake of $\mathbb{D}$.

The collections of weighted geodesics described above are a simple example of discrete measured geodesic laminations of $\mathbb{D}$. The measure of an arc transverse to such a lamination is given by the sum of the weights of the geodesics crossed by the arc (counted according the multiplicity of intersection). Fortunately, if R is of topological type (g, n), then there is a natural topology on the space $\mathfrak{ML}(R)$ of measured geodesic laminations of R in which laminations given by finitely many simple closed geodesics are dense [33].

Lemma 2.7. *If R is a complete hyperbolic surface of type (g, n), then laminations given by finitely many simple closed geodesics are dense in $\mathfrak{ML}(R)$.*

Thus we could have defined earthquakes on such surfaces as the limit of simple earthquakes induced by finite laminations.

2.3. **Shaking our Way Through Teichmüller Space.** It is not difficult to show that every earthquake of $\mathbb{D}$ extends to a homeomorphism of $\partial\mathbb{D}$. Thurston proved that the converse is also true [35]. An excellent exposition of this result is contained in Gardiner and Lakic's recent book [17].

Theorem 2.8. *Every orientation-preserving homeomorphism of $\partial\mathbb{D}$ onto $\partial\mathbb{D}$ can be extended into $\mathbb{D}$ by an earthquake. Moreover, this extension can be approximated uniformly on compact subsets of $\mathbb{D}$ by earthquakes with a finite lamination.*

It is well-known that there is an intimate relationship between metrics on hyperbolic surfaces and homeomorphisms of $\partial\mathbb{D}$. Using Theorem 2.8, Thurston showed that every complete hyperbolic surface can be transformed into any other surface in its Teichmüller space by an earthquake [35].

Thurston's Earthquake Theorem. If R is a complete hyperbolic surface, then any two points in the Teichmüller space of R differ by a unique left earthquake.

The first published proof appears as an appendix to Kerckoff's proof of the Nielsen realization problem [21] and covered only compact surfaces. Thurston's second proof [35] holds for all complete hyperbolic surfaces.

3. Circle Packings

3.1. **Basic Definitions.** Since Thurston's conjecture that circle packings could be used to approximate conformal maps [34], the connections between packings and function theory have been widely studied. In order to make this exposition as self-contained as possible, we will briefly outline the critical definitions and results. More detailed information is contained in the rapidly expanding literature, including several excellent survey articles [14, 23, 30, 32].

Definition 3.1. A **CP-complex** $\mathcal{K}$ is an abstract simplicial 2-complex such that

(1) $\mathcal{K}$ is simplicially equivalent to a triangulation of an (orientable) surface.
(2) Every boundary vertex of $\mathcal{K}$ has an interior neighbor.
(3) The collection of interior vertices is nonempty and edge-connected.
(4) There is an upper bound on the degree of vertices in $\mathcal{K}$.

Notice that a CP-complex is a purely combinatorial object. It possesses no geometric structure until it is embedded in a surface by a circle packing. To emphasize this fact, we will often refer to a CP-complex simply as an **abstract triangulation**. Indeed, the restrictions imposed by conditions 2 through 4 are extremely mild and are met by any reasonable triangulation.

Definition 3.2. A **circle packing** is a configuration of circles with a specified pattern of tangencies. In particular, if $\mathcal{K}$ is a CP-complex, then a circle packing P for $\mathcal{K}$ is a configuration of circles such that

(1) P contains a circle $\mathcal{C}_v$ for each vertex v in $\mathcal{K}$,
(2) $\mathcal{C}_v$ is externally tangent to $\mathcal{C}_u$ if $[v,u]$ is an edge of $\mathcal{K}$,
(3) $\langle \mathcal{C}_v, \mathcal{C}_u, \mathcal{C}_w \rangle$ forms a positively oriented mutually tangent triple of circles if $\langle v,u,w \rangle$ is a positively oriented face of $\mathcal{K}$.

A packing is called **univalent** if none of its circles overlap, that is, if no pair of circles intersect in more than one point.

A univalent circle packing produces a geometric realization of its underlying complex. Vertices can embedded as centers of their corresponding circles, and edges can be realized as geodesic segments joining centers of circles. The collection of triangles embedded in the way is called the **carrier** of the packing, written carr P.

The connection between circle packings and function theory now arises in the investigation of maps between the carriers of two different packings for the same abstract complex. That is, suppose P and $\widetilde{P}$ are both euclidean circle packings for the same underlying complex $\mathcal{K}$. Then every face in $\mathcal{K}$ is realized as both a euclidean triangle T in carr P and a triangle $\widetilde{T}$ in carr $\widetilde{P}$. We can construct an affine map between these realizations by sending each point in T to the point in $\widetilde{T}$ having the same barycentric coordinates. Maps constructed in this way are called **discrete conformal maps**.

3.2. **Existence and Normalization of Packings.** Forcing geometric objects into a pre-defined pattern is a very messy, very nonlinear problem. Indeed, it is not clear from the definition that circle packings should exist in any great number. Fortunately, packings do exist for any CP-complex, and are relatively easy to compute using Stephenson's `CirclePack` program [13,14]. The first existence result, for triangulations of the sphere, was developed independently by Koebe, Andreev, and Thurston [1,2,22,33].

Theorem 3.3. *Every abstract triangulation of the sphere can be realized as a univalent circle packing. This packing is unique up to Möbius transformations.*

It follows immediately that every abstract triangulation $\mathcal{K}$ of a closed disc can be realized as a packing in $\mathbb{D}$. The standard construction [33] adds a single vertex v_∞ to $\mathcal{K}$ with edges to every boundary vertex (that is, we take a simplicial cone over the boundary). The resulting spherical triangulation has a packing on the sphere which we normalize so that $\mathcal{C}_{v_\infty}$ is centered at the north pole and lies on the equator. A packing for the original complex $\mathcal{K}$ now lies in the "bottom half" of the sphere and under stereographic projection becomes a packing in $\mathbb{D}$. Notice that boundary circles will be horocycles, circles internally tangent to $\mathbb{D}$.

There is an alternate construction which for our purposes presents several advantages. Instead of attaching a single vertex to $\mathcal{K}$, we instead form a copy $\mathcal{K}^*$ with the opposite orientation and attach it to $\mathcal{K}$ by identifying corresponding boundary edges. The resulting **combinatorial double** $\mathcal{K}^2 = \mathcal{K}^* \cup \mathcal{K}$ is now a spherical triangulation and can be realized as a packing on the sphere. This packing can be normalized so that boundary vertices of $\mathcal{K}$ lie on equator. Stereographic projection then yields two packings for $\mathcal{K}$. The "bottom half" of the sphere projects to a packing P in $\mathbb{D}$ for which boundary circles are perpendicular to $\partial\mathbb{D}$. The "top half" projects to a packing P^* for $\mathcal{K}^*$. See Figure 2.

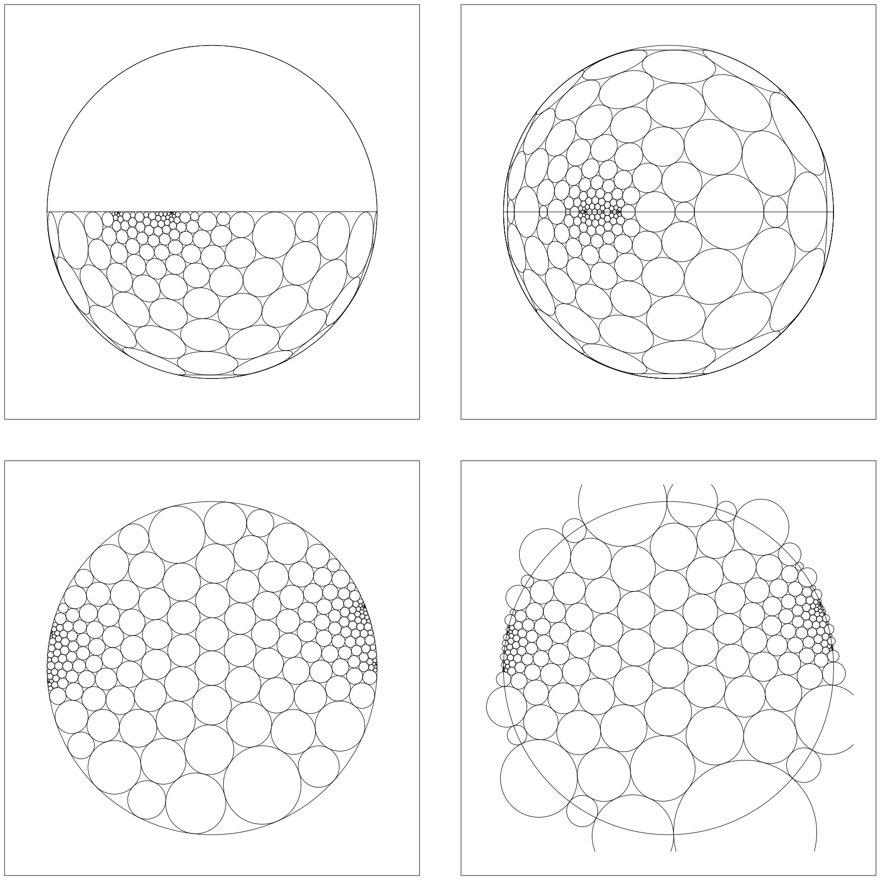

FIGURE 2. The two methods of packing the disk. On the left, one vertex has been added, packed on the sphere (top left), and then projected to the plane (bottom left). Notice that the boundary circles are all tangent to the boundary of unit disk. On the right, the complex has been doubled, packed on the sphere (top right), and then projected to the plane. Notice that the boundary circles are all perpendicular to the boundary of the unit disk.

One advantage to this construction is that the carrier of P completely covers $\mathbb{D}$. A much more significant consequence is that $\operatorname{carr} P^*$ provides a second embedding of $\mathcal{K}$ in $\mathbb{C} \cup \{\infty\}$. As a result, discrete conformal maps defined using P can be extended to (almost) the entire plane by simply repeating the map on P^*. Since triangles of $\mathcal{K}$ are embedded in $\mathbb{C} \cup \{\infty\}$, one must take care in handling any triangles containing infinity. Such triangles do not have euclidean barycentric coordinates, and thus present difficulty in defining discrete conformal maps. We implicitly exclude any such triangles from the domain and range of any discrete conformal maps we construct.

3.3. **Density of Packable Surfaces.** Of course many more surfaces support packings than just the sphere. Existence results for surfaces were given by Thurston [33], Minda and Rodin [26], and Beardon and Stephenson [3]. Moreover, Brooks [9] showed that compact packable surfaces are dense in Teichmüller space. His result was extended to surfaces of finite analytic type by Bowers and Stephenson [6, 7].

Theorem 3.4. *Let $\mathcal{K}$ be an abstract triangulation of a hyperbolic surface of type (g, n, m). Then there exists a unique surface in moduli space which supports a packing for $\mathcal{K}$. A complex along with a choice of marking then determines a unique point in Teichmüller space. Moreover, the collection of all packable surfaces is dense in Teichmüller space.*

In Section 6, we show that combinatorial earthquakes of a single complex can be used to generate a dense packable subset of Teichmüller space.

3.4. **Geometric Properties of Packings.** The interplay between the combinatorial and geometric properties of circle packings gives the theory much of its power. One important example of the influence of the global combinatorics on the geometry of the packing is the Length-Area Lemma of Rodin and Sullivan [29] and the related Spherical Length-Area Lemma [38].

Definition 3.5. A **chain** of circles in packing P for $\mathcal{K}$ is a collection of circles $\mathcal{C}_{v_1}$, $\mathcal{C}_{v_2}, \ldots, \mathcal{C}_{v_n}$ of P so that v_i and v_{i+1} share an edge in $\mathcal{K}$, $i = 1, 2, \ldots, n-1$, and $v_i \neq v_j$, if $i \neq j$. Thus a chain describes a non-self-intersecting edge path in $\mathcal{K}$. A chain is **closed** if $v_1 = v_n$.

Length-Area Lemma. Let P be a univalent circle packing in $\mathbb{D}$ and $\mathcal{C}_v$ a circle in P with euclidean radius r. Assume there are m disjoint chains of circles in P having combinatorial lengths $n_1, n_2, \ldots, n_m$, and each separating $\mathcal{C}_v$ from the origin and a point on the boundary. Then

$$r \leq \frac{1}{\sqrt{\sum_{i=1}^{m} \frac{1}{n_i}}}.$$

Spherical Length-Area Lemma. Let P be a univalent circle packing on the Riemann sphere and $\mathcal{C}_v$ a circle in P with spherical radius r. Assume there exist m closed disjoint chains of circles in P having combinatorial lengths $n_1, n_2, \ldots, n_m$, and each separating $\mathcal{C}_v$ from two points q_1 and q_2. If the spherical distance from q_1 to q_2 is d, then

$$r < \frac{2\delta\pi}{\sqrt{\sum_{i=1}^{m} \frac{1}{n_i}}} \max\left\{1, \frac{\pi - d}{d}\right\},$$

where $\delta = \frac{\pi - d}{\pi \sin(\pi - d)}$.

The Length-Area Lemmas can be interpreted as stating that circles in "large packings" must be "small." Thus purely combinatorial considerations can be used to force the mesh of packings to 0.

The local combinatorics of a packing also exert an important influence. This observation led to Rodin and Sullivan's Ring Lemma [29], the first connection between combinatorial properties of packings and the and function theoretic properties of associated discrete conformal maps.

Ring Lemma. If v an interior vertex of $\mathcal{K}$ and w is a neighbor of v, then there is a lower bound C_D on the ratio of the radius of $\mathcal{C}_w$ to $\mathcal{C}_v$ in any univalent packing for $\mathcal{K}$. The bound depends only on the degree D of v.

It is easy to show that affine maps between triangles are quasiconformal with dilatation depending only on the difference in corresponding angles of the triangles. The Ring Lemma implies that angles at interior vertices in the carrier of a univalent packing are bounded away from 0 and π. This gives an upper bound on the difference between corresponding angles in the carriers of two different packings for the same complex. Consequently, discrete conformal maps are k-quasiconformal on faces which do not contain a boundary vertex. The constant k depends only on the degree of the packing.

3.5. **Discrete Function Theory.** Of course, the great interest in discrete conformal maps is that they are not only quasiconformal, but (as the name suggests) almost conformal. This connection was first conjectured by Thurston [33] for degree 6 packings and was proven by Rodin and Sullivan [29].

Rodin-Sullivan Theorem. Fix a simply connected domain $\Omega \subsetneq \mathbb{C}$ and points $p, q \in \Omega$. Let P_n be the portion of the infinite regular degree 6 packing which intersects Ω and $\mathcal{K}_n$ its underlying complex. Let $\widetilde{P}_n$ be a packing in $\mathbb{D}$ for $\mathcal{K}_n$ with all boundary circles tangent to $\partial\mathbb{D}$, and let $f_n : \operatorname{carr} P_n \to \operatorname{carr} \widetilde{P}_n$ be the induced discrete conformal map. If each $\widetilde{P}_n$ has been normalized so that $f_n(p) = 0$ and $f_n(q) > 0$, then $\{f_n\}$ converges locally uniformly to the Riemann map $f : \Omega \to \mathbb{D}$ satisfying $f(p) = 0$ and $f(q) > 0$.

Stephenson [31] relaxed the degree 6 condition using random walk techniques, and He and Rodin [18] showed that only a uniform bound on the degree was necessary.

Theorem 3.6. *If a circle C_v is surrounded by n chains of circles, each circle having degree at most m, then the dilation of a discrete conformal map defined on faces containing v decreases to 0 as $n \to \infty$. In particular, the Rodin-Sullivan Theorem holds for packings of uniformly bounded degree.*

3.6. **Hex Refinement.** For our approximation results, we will need a method of refining our packings. Any scheme which maintains uniform control on the degree of the complexes will do; however, the hex refinement method of Bowers and Stephenson [5] is especially nice.

Definition 3.7. If $\mathcal{K}$ is a CP-complex, the **hex refinement** of $\mathcal{K}$ is the complex formed by adding a vertex to each edge and adding an edge between new vertices lying on the same face. See Figure 3.

Proposition 3.8. *Any new interior vertices added to $\mathcal{K}$ by hex refinement have degree 6, while the degree of original vertices remain unchanged. If $\mathcal{K}$ is embedded in $\mathbb{C}$ in such a way that edges correspond to euclidean line segments, then its hex refinement may be realized by adding line segments joining the mid-point of each edge. In this case, each face of $\mathcal{K}$ will be subdivided into four new faces, each similar to the original and having edges one-half as long.*

Notice that if we refine only one face in the complex, then we will cease to have a triangulation. The faces bordering the refined face will have an extra vertex, making them combinatorial quadrilaterals instead of combinatorial triangles. To rectify this situation, we add a new edge connecting the extra vertex to the opposite vertex. See Fig 4. Notice that the angles formed by drawing a line segment from a vertex to the midpoint of the opposite side depend only on the original angles. Thus if the original angles were bounded away from 0 and π, we may embed our

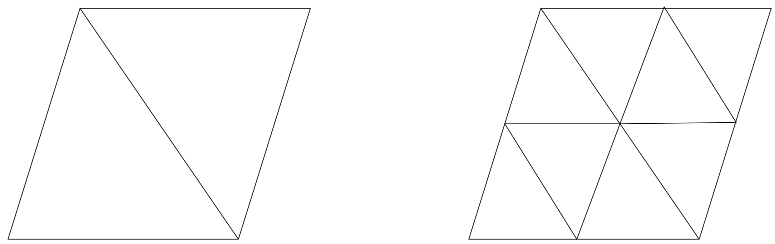

FIGURE 3. Two triangles before (left) and after (right) hex refinement. Notice that the interior vertex has degree 6

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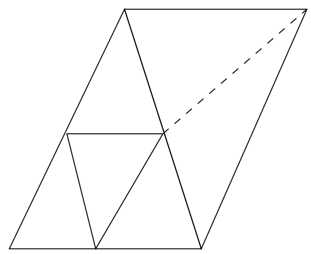

FIGURE 4. Hex refinement of just one triangle requires adding an edge to adjacent triangles.

refinement as above and be assured that angles in the new complex will be also be bounded.

Of course, the trick of adding edges to absorb extra vertices can be used to hex refine large pieces of a complex, not just one face. As a result, we can refine a complex locally to improve our discrete approximations in troublesome regions.

4. Combinatorial Constructions

4.1. **Cutting and Re-attaching.** Next we will describe a generalization of the combinatorial welding technique developed in [38]. Suppose abstract triangulations $\mathcal{K}_1$ and $\mathcal{K}_2$ are embedded in a surface S and $h : B_1 \subset \partial\mathcal{K}_1 \to B_2 \subset \partial\mathcal{K}_2$ is a homeomorphism. We will use the map h to attach $\mathcal{K}_1$ to $\mathcal{K}_2$ along B_1 and B_2. In Section 5, h will be taken to be a discretized comparison isometry between strata of a measured geodesic lamination, and our attaching will act as a combinatorial earthquake.

If h respects the combinatorial structures of B_1 and B_2, sending vertices to vertices and edges to edges, then we can simply identify these vertices and edges with their images under h. In general, however, our identification maps will not be so well-behaved. As a result, it will be necessary to slightly modify $\mathcal{K}_1$ and $\mathcal{K}_2$ so that h respects their new combinatorial structures.

Specifically, for each boundary vertex v of B_1 we will add a vertex to B_2 embedded at the point $h(v)$. Similarly, we add a vertex at $h^{-1}(w)$ for every vertex w of B_2.

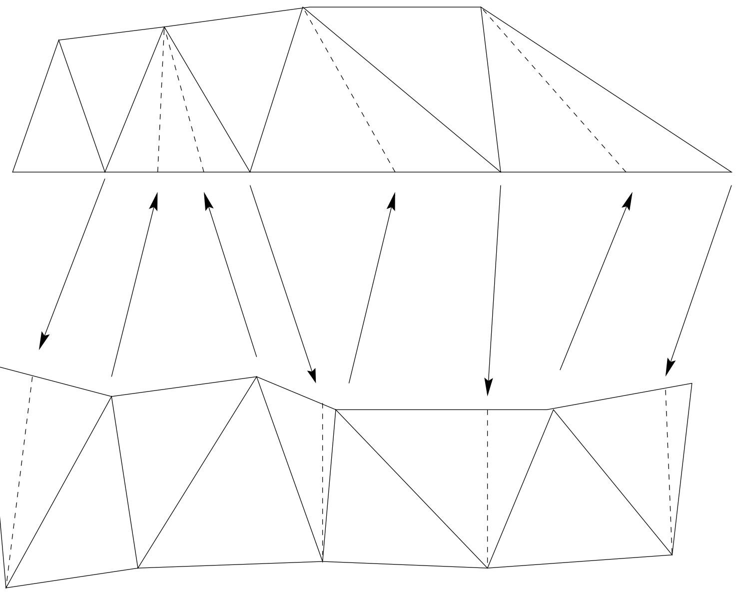

FIGURE 5. The results of modifying $\mathcal{K}_1$ and $\mathcal{K}_2$ so that h respects their combinatorial structures. The arrows represent the images of original boundary vertices under h and h^{-1}. The dashed lines indicate the location of new edges.

Of course, we can't go about adding vertices at will and expect that $\mathcal{K}_1$ and $\mathcal{K}_2$ will remain triangulations. Thus if we add a vertex $h(v)$ or $h^{-1}(w)$ to an edge $[a, b]$ in a triangle $< a, b, c >$, then we also add an edge from c to the new vertex. In this way, we augment the original complexes K_1 and $\mathcal{K}_2$ so that h respects their new combinatorial structures. See Figure 5.

Notice that the second condition in the definition of a CP-complex implies each face can have at most one edge contained in the boundary. Thus no vertices will be added to another edge, for example $[b, c]$, that will interfere will the edges just created.

4.2. **Controlling Geometry.** The real action here lies in the interplay between the combinatorics of the complex and the geometry produced by the corresponding circle packing. A complex $\mathcal{K}$ formed by attaching $\mathcal{K}_1$ and $\mathcal{K}_2$ as described above can be packed and embedded nicely. In particular, the Ring Lemma implies angles will be bounded away from 0 if the degree is bounded.

If $\mathcal{K}_1$ and $\mathcal{K}_2$ themselves were embedded in $\mathbb{C}$ using a circle packing, then the Ring Lemma also gives bounds on the angles in $\operatorname{carr} K_1$ and $\operatorname{carr} K_2$. Thus we could construct quasiconformal discrete maps as described in section 3.5. We need only ensure

1. (1) The degree of the new complex $\mathcal{K}$ is bounded.
2. (2) The new edges added to $\operatorname{carr}\mathcal{K}_1$ and $\operatorname{carr}\mathcal{K}_2$ do not result in triangles with arbitrarily small angles.

If $\mathcal{K}_1$ and $\mathcal{K}_2$ are finite, then conditions 1 and 2 clearly hold. However, the bounds depend on the particular complexes $\mathcal{K}_1$ and $\mathcal{K}_2$. If we create a sequence glued complexes, we would like to have bounds which hold throughout the sequence. In particular, if the complexes all have degree less than D and the attaching maps are k-bilipschitz, then our bounds should depend only on D and k.

Thus we must take a bit more care when adding vertices and gluing. We must also assume that edges in B_1 are roughly the same size as the edges in their image in B_2. That is, we require a uniform constant C so that for each $[v,w]$ in B_1

$$\frac{1}{C}|v-w| \leq |a-b| \leq C|v-w| \tag{4.1}$$

whenever $[a,b]$ is an edge in B_2 intersecting $h([v,w])$. The corresponding condition should also hold between edges of B_2 and their images under h^{-1} in B_1.

If (4.1) does not hold, we can easily rectify the situation by breaking up any edges which are too long using repeated local hex refinement.

Now notice that the number of vertices in $h^{-1}([v,w])$ is at most

$$\frac{\text{length of } h^{-1}([v,w])}{\min\{|a-b| : [a,b] \cap h([v,w]) \neq \emptyset\}} \leq \frac{k|v-w|}{\frac{1}{C}|v-w|} = kC.$$

Thus at most kC vertices can be added the edge $[v,w]$ during our gluing process. Similarly, at most kC vertices can be added to any edge $[a,b]$ of B_2. Consequently, condition 1 above must hold.

Condition 2 will hold if we ensure that no vertex is added too near an existing vertex and that no two vertices are added too near each other. But notice that if $h([a,b]) \subset [v,w]$, then

$$|h(a)-h(b)| \geq \frac{1}{k}|a-b| \geq \frac{1}{kC}|v-w|. \tag{4.2}$$

If some vertex $h(a)$ lies within $\frac{1}{kC}|v-w|$ of v, then we can modify our gluing to "round" $h(a)$ off to v. By (4.2) we see no two vertices $h(a)$ and $h(b)$ will be rounded off to the same v.

Thus any vertices $h(a)$ and $h(b)$ added to $[v,w]$ will be separated from v, w, and each other by at least $\frac{1}{kC}|v-w|$. It is now an elementary exercise in euclidean geometry to show that condition 2 must hold.

Proposition 4.1. *Combinatorial attachment by h induces a piecewise linearization $\widehat{h}$ of h. If h is bilipschitz, then the linearization will be as well.*

Proof. Since our combinatorial augmentations ensured that h sends vertices of B_1 to vertices of B_2, we set $\widehat{h}(v) = h(v)$ on the vertices. If the edges $[v,w]$ and $[h(v),h(w)]$ are parameterized proportionally to arc length by $\alpha : [0,1] \to [v,w]$ and $\beta : [0,1] \to [h(v),h(w)]$, respectively, then we define $\widehat{h}$ on $[v,w]$ by

$$\widehat{h}(\alpha(t)) = \beta(t).$$

Notice that if the the edges are embedded as euclidean line segments, $\widehat{h}$ is piecewise linear. A straightforward calculation shows that $\widehat{h}$ is bilipschitz [38]. □

5. Approximating Earthquakes

5.1. **Combinatorial Laminations.** Throughout this section, let $\mathcal{L}$ be a fixed finite measured geodesic lamination of $\mathbb{D}$ with geodesics $\{L^{i,j}\}$. We will assume the geodesics are ordered so that stratum S^i lies on the left of $L^{i,j}$ while S^j lies on its right. We can also choose our ordering so that S^1 is bounded on one side by an arc of $\partial\mathbb{D}$ and S^1 and S^2 are adjacent strata. We denote the endpoints of $\ell^{1,2}$ by A and B.

We saw in Section 2 that $\mathcal{L}$ induces an earthquake map E with comparison isometries

$$H^{i,j} = (E|_{S^i})^{-1} \circ E|_{S^j}.$$

Notice that if M is a hyperbolic isometry, then $E = M^{-1}EM$; hence we may assume E fixes the endpoints A and B of $L^{1,2}$.

Let $\{P_n\}$ be a sequence of finite packings in $\mathbb{D}$ with uniformly bounded degree and mesh decreasing to 0. As described in Section 3.2, we may assume $\{P_n\}$ was obtained as the "bottom half" of some packing $P_n^2 = P_n \cup P_n^*$ for a spherical triangulation $K_n^2 = K_n \cup K_n^*$. By scaling slightly and possibly rotating P_n^2, we may assume for sufficiently large n that circles two circles $\mathcal{C}_A$ and $\mathcal{C}_B$ are centered at A and B, respectively.

We let D_n denote the euclidean carrier of P_n, and D_n^* denote the euclidean carrier of P_n^* minus any triangles containing infinity. Notice that D_n does not equal $\mathbb{D}$, nor is it even completely contained in $\mathbb{D}$. Working in our discrete universe, such complications will be common, but will fortunately disappear in the limit. For example, it is clear that as $n \to \infty$,

$$D_n \to \mathbb{D}$$
$$D_n^* \to \mathbb{D}^*,$$

where $\mathbb{D}^* = \mathbb{C} \setminus \overline{\mathbb{D}}$.

A similar complication arises when we attempt to shear D_n along the geodesics of our lamination. In general, each $L^{i,j}$ will pass indiscriminately through triangles and vertices in D_n with no regard for the underlying combinatorial structure. Our discretized $L^{i,j}$ must therefore be a rather jagged edge path connecting those vertices close to $L^{i,j}$. In this section we construct such a **combinatorial lamination** $\{\ell_n^{i,j}\}$ of D_n corresponding to $\{L^{i,j}\}$.

If a_n^i and b_n^i are the boundary vertices of D_n embedded most closely to the endpoints of $L^{i,j}$, we would like take as our discrete version of $L^{i,j}$ the simple edge path $\ell_n^{i,j}$ connecting a_n^i and b_n^i whose Hausdorff distance to $L^{i,j}$ is minimal. However, to avoid certain technical difficulties, we will also require $\ell_n^{i,j}$ satisfy the following:

(1) Every vertex in $\ell_n^{i,j}$ must have a neighbor in both components of $D_n \setminus \ell_n^{i,j}$.
(2) The $\ell_n^{i,j}$'s are disjoint, except possibly at their common endpoints.
(3) Any hyperbolic geodesic perpendicular to $L^{i,j}$ hits $\ell_n^{i,j}$ only once.

Lemma 5.1. *By taking n sufficiently large and possibly modifying D_n near $L_n^{i,j}$, we can find paths $\ell_n^{i,j}$ so that conditions 1 through 3 are satisfied. Moreover, $\ell_n^{i,j} \to L^{i,j}$ uniformly as $n \to \infty$.*

Proof. First notice that if the minimal path contains two edges of the same face, we may modify the path by replacing these two edges by the third edge of the face. Thus we may assume $\ell_n^{i,j}$ contains at most one edge from each face.

But now observe that if $[a,b] \subset \ell_n^{i,j}$ is the common edge of faces $< a,b,c >$ and $< a,b,d >$, then c and d lie in different components of $D_n \setminus \ell_n^{i,j}$. Hence condition 1 is satisfied.

Furthermore, since the mesh of D_n decreases to 0, we can take n large enough to fulfill condition 2.

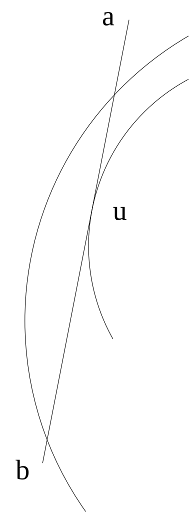

FIGURE 6. An edge which is hit twice by geodesics perpendicular to $L^{i,j}$.

Now recall that any geodesic η perpendicular to $L^{i,j}$ is a portion of a circle and thus can intersect one of the euclidean line segments which comprise the edge path $\ell_n^{i,j}$ at most twice. Unfortunately, it is entirely possible that an edge will be hit twice by some geodesic. In fact, if $[a, b]$ is hit twice by any perpendicular geodesic η, it will be hit twice by each member of an infinite collection of such geodesics. This collection will be bounded on one side by a geodesic hitting either a or b and on the other side by a geodesic which hits $[a, b]$ in exactly one point u. See Figure 6.

Notice that u divides $[a, b]$ into two segments, each of which satisfies condition 3. Thus breaking the edge $[a, b]$ by adding a new vertex at u would ensure condition 3. However, if u is very near either a or b, we would loose the uniform bound on the angles in triangles of D_n otherwise promised by the Ring Lemma. Consequently, let us fix $0 < \epsilon << \frac{1}{2}$ independently of i and n and keep ϵ fixed throughout the construction. If the distance from u to a or from u to b is less than $\epsilon|a - b|$, then we will move a or b, respectively, to u. This perturbation at the boundary is another unfortunate side-effect of working with discrete objects in a continuous world, but it can be made arbitrarily small at the cost of slightly greater quasiconformal distortion at the boundary.

After each such perturbation is made, the location of the breaking point on a neighboring edge may have changed. Thus after each of the perturbations, we will recalculate the breaking points and re-label as necessary. If we begin making our perturbations on some edge near the middle of $\ell_n^{i,j}$ and work our way simultaneously toward the boundary, we can be assured each new perturbation will not disturb any previously visited edges.

After all perturbations have been made, we will add a vertex at the remaining breaking points. If $[a, b]$ is again an edge with breaking point u and $[a, b]$ is the common edge of faces $< a, b, c >$ and $< a, b, d >$, then we will add edges from u to c and d. Thus D_n will remain a triangulation and, though the bound may have slightly changed, will still have angles bounded away from 0 and π uniformly in n.

Finally note that none of our modifications will move $\ell_n^{i,j}$ from its original location a distance greater than the mesh of D_n. Thus as $n \to \infty$, we see $\ell_n^{i,j} \to L^{i,j}$ uniformly. $\square$

5.2. **Hyperbolic Projections.** Next we need to develop a discrete analog of the hyperbolic shearing maps $h^{i,j}$. Since the polygonal approximations $\ell_n^{i,j}$ to $L^{i,j}$ are not invariant under $H^{i,j}$, we cannot use the $H^{i,j}$ directly. Instead, we will form hyperbolic projections $p_n^{i,j}$ which will map $\ell_n^{i,j}$ to $L^{i,j}$.

Notice that the collection of hyperbolic geodesics perpendicular to $L^{i,j}$ completely fill $\mathbb{D}$. By condition 3 of Lemma 5.1, each perpendicular geodesic η hits each $\ell_n^{i,j}$ only once. We can then define a map $p_n^{i,j} : D_n \cap \mathbb{D} \to \mathbb{D}$ by the requirement that $p_n^{i,j}|\eta$ be the unique hyperbolic Möbius transformation with axis η and translation length equal to the hyperbolic distance between $\ell_n^{i,j} \cap \eta$ and $L^{i,j}$.

Proposition 5.2. *The sequence $\{p_n^{i,j}\}$ is uniformly bilipschitz on $\overline{\mathbb{D}}$ and converges uniformly to the identity map on $\overline{\mathbb{D}}$.*

Proof. Note that the distance between $\ell_n^{i,j}$ and $L^{i,j}$ is a function of the mesh of D_n. Since each $p_n^{i,j}$ moves no point more than this distance, it must be bilipschitz. As $n \to \infty$, the mesh of D_n converges to 0, so the bilipschitz constant must converge to 1. $\square$

5.3. **Discrete Shearing Maps.** Now we use our projections to "fill in the gaps" between each $\ell_n^{i,j}$ and $L^{i,j}$.

If $H^{i,j}$ is the the comparison Möbius transformation with axis $L^{i,j}$ and translation length determined by the transverse measure on $\mathcal{L}$, then the map

$$(p_n^{i,j})^{-1} \circ H^{i,j} \circ p_n^{i,j} \tag{5.1}$$

acts on $(p_n^{i,j})^{-1}(L^{i,j}) \subset \ell_n^{i,j}$.

Moreover, since the maps $p_n^{i,j}$ were uniformly bilipschitz by Proposition 5.2 and each $H^{i,j}$ is conformal, the maps $(p_n^{i,j})^{-1} \circ H^{i,j} \circ p_n^{i,j}$ are bilipschitz. Hence the method of Section 4 can be used to combinatorially shear D_n along $\ell_n^{i,j}$. We will denote by

$$h_n^{i,j} : \ell^{i,j} \to \ell^{i,j}$$

the piecewise linear discretized map resulting from Proposition 4.1.

Remark 5.3. It may well be that the image of an endpoint of $\ell_n^{i,j}$ under $(p_n^{i,j})^{-1} \circ H^{i,j} \circ p_n^{i,j}$ or its inverse may lie outside of $\ell_n^{i,j}$. In such a case, we will adjust our gluing on the first and last edges of $\ell_n^{i,j}$ so that the endpoints are always attached.

Proposition 5.4. *The discretized maps $h_n^{i,j}$ are bilipschitz with bilipschitz constants bounded independently of i and n. More importantly, each can be extended to all of $\mathbb{D} \cup D_n$ in such a way that $h_n^{i,j}$ converges uniformly to $H^{i,j}$ on $\mathbb{D}$.*

Proof. The first assertion follows from Proposition 4.1.

Since each geodesic η perpendicular to $L^{i,j}$ hits $\ell_n^{i,j}$ only once, we can extend the action of $h_n^{i,j}$ to all of $\mathbb{D}$ by using a hyperbolic Möbius transformation to send each η to the perpendicular geodesic through $h_n^{i,j}(\eta \cap \ell_n^{i,j})$. On $D_n \setminus \mathbb{D}$, any reasonable extension will suffice for our purposes (recall that $D_n \setminus \mathbb{D}$ will disappear as $n \to \infty$). For example, since each perpendicular η is a portion of a circle which does extend outside $\mathbb{D}$, we could continue to use these perpendicular geodesics to extend $h_n^{i,j}$.

Now it is clear from the construction that $h_n^{i,j}$ uniformly approximates $\left(p_n^{i,j}\right)^{-1} \circ H^{i,j} \circ p_n^{i,j}$ on $\ell_n^{i,j}$, and $\left(p_n^{i,j}\right)^{-1} \circ H^{i,j} \circ p_n^{i,j}$ converges to $H^{i,j}$ uniformly by Proposition 5.2. Notice that $H^{i,j}$ is characterized by taking geodesics perpendicular to $L^{i,j}$ at x to perpendicular geodesics through $H^{i,j}(x)$. But this is precisely the behavior of $h_n^{i,j}$ in the limit. □

Repeatedly shearing and re-attaching D_n along the **combinatorial lamination** $\{\ell_n^{i,j}\}$ produces a new complex $\widetilde{\mathcal{K}}_n$, the result of a **combinatorial earthquake** of $\mathcal{K}_n$. We form the double $\widetilde{\mathcal{K}}_n^2 = \widetilde{\mathcal{K}}_n \cup \widetilde{\mathcal{K}}_n^*$ of $\widetilde{\mathcal{K}}_n$. By Theorem 3.3, there is a packing for $\widetilde{\mathcal{K}}_n^2$ on the Riemann sphere normalized so that the boundary edges between $\widetilde{\mathcal{K}}_n$ and $\widetilde{\mathcal{K}}_n^*$ lie on $\partial\mathbb{D}$. Stereographically projecting this packing yields a euclidean packing $\widetilde{P}_n^2 = \widetilde{P}_n \cup \widetilde{P}_n^*$ (ignoring infinity for now). We may scale and rotate $\widetilde{P}_n^2$ so that the vertices a and b corresponding in D_n to the endpoints A and B of $L^{1,2}$ are once again placed at A and B, respectively.

We let $\widetilde{D}_n$ be the euclidean carrier of $\widetilde{P}_n$ and $\widetilde{D}_n^*$ be the euclidean carrier of $\widetilde{P}_n^*$ minus any triangles containing infinity.

Proposition 5.5. *As $n \to \infty$, then $\widetilde{D}_n \to \mathbb{D}$ and $\widetilde{D}_n^* \to \mathbb{D}^*$.*

Proof. The use of the Length-Area Lemmas to show that the radii of circles in the range of a discrete conformal map decrease to 0 is by now standard in the circle packing literature [18, 29, 31, 37, 38]. As $n \to \infty$, any circle in $\widetilde{P}_n^2$ can be separated from A and B by an ever-increasing number of chains. The Spherical Length-Area Lemma can then be used to force the spherical radius to 0 as $n \to \infty$.

Since boundary circles of $\widetilde{P}_n$ and $\widetilde{P}_n^*$ were centered on $\partial\mathbb{D}$ before being stereographically projected to $\mathbb{C}$, their euclidean and spherical radii are uniformly comparable. Hence the euclidean radii must decrease to 0 as well. But no point of D_n can extend further outside of $\mathbb{D}$ than the maximum radius of the boundary circles of $\widetilde{P}_n$. Thus $D_n \to \mathbb{D}$ as $n \to \infty$, and similarly, $\widetilde{D}_n^* \to \mathbb{D}^*$. □

5.4. **Discrete Earthquake Maps.** Finally, we can use our circle packings to define **discrete earthquake maps**

$$E_n : D_n \to \widetilde{D}_n$$

by sending triangles to triangles as in Section 3.5.

In keeping with our previous notation, let S_n^i be the i^{th} combinatorial stratum of D_n, that is, the subset of D_n bounded by $\{\ell^{i,j}\} \cup \{\ell^{k,i}\}$. Similarly, we denote by $\widetilde{S}_n^i$ the corresponding combinatorial stratum of $\widetilde{D}_n$.

The proof that E_n actually converges to the earthquake map E will involve the careful analysis of the action of E_n on each stratum. To simplify notation, we define

$$f_n^i = E_n|_{S_n^i} : S_n^i \to \widetilde{S}_n^i.$$

Furthermore, notice that by our construction we have the discrete version

$$f_n^j = f_n^i h_n^{i,j} \tag{5.2}$$

of (2.1) on $\ell_n^{i,j}$.

5.5. **Extending Beyond the Original Strata.** Next we will recursively extend each f_n^i to (almost) all of $\mathbb{C}$. By Proposition 5.4, each of the gluing maps $h_n^{i,j}$ are defined on all of $\mathbb{D}$. Thus we can extend each f^j across $\ell^{i,j}$ to S^i by setting

$$f_n^j = f_n^i h_n^{i,j} \tag{5.3}$$

on S_n^i. Observe that this extension is continuous by (5.2).

Similarly, we extend each f^i across $\ell^{i,j}$ by setting

$$f_n^i = f_n^j \left(h_n^{i,j}\right)^{-1} \tag{5.4}$$

on S_n^j. Again, this extension is continuous by (5.2).

But notice now that equations (5.3) and (5.4) serve to extend f_n^i and f_n^j not only to neighboring strata but also to all of D_n. For example, suppose stratum S_n^2 intersects strata S_n^1 along $\ell_n^{1,2}$ and S_n^3 along $\ell_n^{2,3}$. Then we define f_n^1 on S_n^2 to be $f_n^2 h_n^{1,2}$. Similarly, we set $f_n^2 = f_n^3 h_n^{2,3}$ on S_n^3. But the extension of f_n^2 extends our extension of f_n^1; that is, we can set

$$f_n^1 = f_n^2 h^{1,2} = f_n^3 h_n^{2,3} h_n^{1,2}$$

on S_n^3.

Similarly, we set

$$f_n^3 = f_n^2 \left(h^{2,3}\right)^{-1} = f_n^1 \left(h^{1,2}\right)^{-1} \left(h^{2,3}\right)^{-1}$$

on S_n^1.

Next observe that because we created the packings P_n and $\widetilde{P}_n$ using the doubling construction of Section 3.2, each combinatorial stratum $S_n^i \subset D_n$ has a copy embedded in D_n^* (ignoring for now any triangles containing infinity). Consequently, we can repeat the action of each f_n^i on D_n^* and thus extend f_n^i to almost all of $\mathbb{C}$.

Of course, we must omit from our scheme any triangles in containing infinity since these are not properly embedded in $\mathbb{C}$. Fortunately, these triangles will disappear in the limit.

Proposition 5.6. *The spherical diameter of the triangles in the carrier of P_n or $\widetilde{P}_n$ which contain infinity decreases to 0 as $n \to \infty$. Consequently, a given compact subset of $\mathbb{C}$ will lie outside these triangles for sufficiently large n.*

Proof. Since the degree of P_n is uniformly bounded, the proposition is an easy consequence of the Spherical Length-Area Lemma. □

5.6. **Proof of Convergence.** A careful study of the extended stratum maps f_n^i now leads to our approximation theorem for earthquakes of the disk.

Theorem 1. Given any finite measured geodesic lamination $\mathcal{L}$ and a sequence of finite bounded degree packings P_n in $\mathbb{D}$ with mesh decreasing to 0, the discrete earthquake maps E_n formed by combinatorial earthquakes of P_n converge locally uniformly to the earthquake map E induced by $\mathcal{L}$.

Proof. First recall that by our numbering scheme, S^1 is bounded on one side by an arc of $\partial\mathbb{D}$. Consequently, S^1 lies on the right side of the other strata. Thus to extend some f^i to S^1, requires repeated applications of (5.4). As a result,

$$f_n^i = f_n^1 \left(h_n^{1,2}\right)^{-1} \cdots \left(h_n^{i,j}\right)^{-1}. \tag{5.5}$$

But since f_n^1 fixes both A and B, f_n^i must fix both $h_n^{i,j}\dots h_n^{1,2}(A)$ and $h_n^{i,j}\dots h_n^{1,2}(B)$. By Proposition 5.4,

$$h_n^{i,j}\dots h_n^{1,2}(A) \to H^{i,j}\dots H^{1,2}(A)$$
$$h_n^{i,j}\dots h_n^{1,2}(B) \to H^{i,j}\dots H^{1,2}(B). \tag{5.6}$$

Since $H^{i,j}\cdots H^{1,2}(A) \neq H^{i,j}\cdots H^{1,2}(B)$, there is a uniform lower bound on the distance between $f_n^i h_n^{i,j}\dots h_n^{1,2}(A)$ and $f_n^i h_n^{i,j}\dots h_n^{1,2}(B)$ (uniform in both n and i). Moreover, the range of each f_n^i also omits the point at infinity. Thus by well-known normality results for quasiconformal mappings [25], each of the sequences $\{f_n^i\}_n$ converges to a quasiconformal map F^i uniformly on compact subsets of $\mathbb{C}$. Since $f_n^i = E_n|_{S_n^i}$, E_n converges uniformly on compact subsets of the strata.

But now Theorem 3.6 [18] implies that F^i is conformal on the limit strata S^i and S^{i^*}. Moreover, since each f_n^i was extended off S^i using maps which converge uniformly to Möbius transformations and $\cup_{i,j}\ell^{i,j}$ has measure 0, F^i must be conformal on all of $\mathbb{C}$. But this implies F^i is itself is Möbius. Since it fixes $\mathbb{D}$, it must be a hyperbolic isometry.

Finally notice that since $f_n^j = f_n^i h_n^{i,j}$ on $\ell_n^{i,j}$,

$$F^j = F^i H^{i,j}$$

or equivalently,

$$H^{i,j} = \left(F^i\right)^{-1} \circ F^j$$

on $L^{i,j}$. Thus the limit of the sequence E_n must be the unique earthquake map E. □

5.7. **Extending Boundary Maps.** By Theorem 2.8 [17, 35], every homeomorphism of $\partial\mathbb{D}$ can be extended to $\mathbb{D}$ by an earthquake. Such a map is the limit of earthquakes with finite laminations. Applying a standard diagonalization argument to Theorem 1, we have the following useful corollary.

Corollary 5.7. *Any orientation-preserving homeomorphism of $\partial\mathbb{D}$ has a continuous extension to $\mathbb{D}$ constructed as a limit of discrete earthquake maps.*

6. Packable Surfaces

6.1. **Discrete Earthquakes on Surfaces.** Now we consider the effect of combinatorial earthquakes on triangulations of surfaces. Suppose $\mathcal{K}_0$ is a CP-complex simplicially equivalent to a triangulation of a hyperbolic surface of type (g, n). Then there is a unique Riemann surface which supports a packing for $\mathcal{K}_0$; that is, the purely combinatorial information encoded in $\mathcal{K}_0$ determines a unique point in Teichmüller space. By deforming $\mathcal{K}_0$ by an appropriate combinatorial earthquake (and hex refinements) we can move to any other point in Teichmüller space.

Theorem 2. Let $\mathcal{K}_0$ be a marked abstract triangulation of a surface of topological type (g, n). Then $\mathcal{K}_0$ generates a dense packable subset of Teichmüller space by hex refinement and combinatorial earthquakes. That is, for any Riemann surface R of type (g, n) there is a sequence of combinatorial earthquakes and refinements of $\mathcal{K}_0$ which produce packable surfaces converging to R.

Proof. Let $\mathcal{K}_n$ be the complex formed by hex refining $\mathcal{K}_0$ n times. Then there exists a unique Riemann surface R_n which supports a packing P_n for $\mathcal{K}_n$ [3].

The circle packings induce maps from R_0 to each R_n as follows: Lift P_0 and P_n to the universal cover $\mathbb{D}$. The carrier of P_n provides an embedding of $\mathcal{K}_n$. Notice however, that $\mathcal{K}_n$ can also be embedded by refining $\operatorname{carr} P_0$ n times as described in Proposition 3.8. We can construct a discrete conformal map f_n between these embeddings sending triangles to triangles. By the Ring Lemma [29] this map will be K-quasiconformal with K depending only on the degree of $\mathcal{K}$; in particular, K will be independent of n.

Now recall that the metric in Teichmüller space was defined between quasiconformal images of some base surface. If we select R_0 as our base surface, then R_n is the image of R_0 under f_n and R_0 is the image of itself under the identity map. Thus the Teichmüller distance between R_0 and R_n is given by

$$\frac{1}{2}\log K^*$$

where $K*$ is the minimal dilatation of $g_2 \circ (g_1)^{-1}$ for quasiconformal maps g_1 homotopic to the identity on R_0 and g_2 homotopic f_n on R_n. In particular, since f_n and the identity qualify as test functions, the distance between R_n and R_0 is bounded by

$$\frac{1}{2}\log(K).$$

Thus for each n, R_n lies inside a ball of radius $\frac{1}{2}\log K$ centered at R_0. The closure of this ball is compact; hence the sequence $\{R_n\}$ has a subsequence converging to a surface R. For simplicity of notation, we will re-number and assume the entire sequence converges to R. (In fact, it follows from the work of Bowers and Stephenson [5], that the entire sequence converges to the point R determined by gluing together euclidean equilateral triangles in the pattern given by $\mathcal{K}_0$).

Now let S be any other point in the Teichmüller space of R. By Theorem 2.3 [35], there is a measured geodesic lamination $\mathcal{L}$ of R and a corresponding earthquake E which transforms R into S. Recall that by Lemma 2.7, E is the limit of a a sequence $\{E_m\}$ of earthquakes whose laminations $\{\mathcal{L}_m\}$ consist of finitely many closed curves [21]. We can transfer the lamination corresponding to E_m to a lamination $\mathcal{L}_{m,n}$ of R_n by the Teichmüller maps T_{R,R_n}. The images of the laminations themselves might not be geodesic laminations of R_n, but they can be straightened to a finite geodesic lamination. Recall that the convergence of R_n to R is equivalent to the convergence of the Teichmüller maps T_{R,R_n} from R to R_n to a conformal map of R homotopic to the identity. Consequently, as $n \to \infty$, the lifts of $\widetilde{\mathcal{L}}_{m,n}$ to $\mathbb{D}$ converge in $\mathfrak{ML}(\mathbb{D})$ to the lift $\widetilde{\mathcal{L}}_m$ of $\mathcal{L}_m$.

Now each $\mathcal{L}_{m,n}$ induces an earthquake $E_{m,n}$ on R_n Because E and E_m are defined on R and each $E_{m,n}$ is defined on R_n we will lift all these to earthquakes of the disk and consider their effects there. By a slight modification of the proof of Theorem 1, we will see that the combinatorial earthquakes $\widetilde{E}_{n,m}$ created by the lifts $\widetilde{\mathcal{L}}_{m,n}$ of $\mathcal{L}_{m,n}$ converge to the lift $\widetilde{E}_m$ of E_m as $n \to \infty$.

First note that the existence of the original packings P_n on R_n was shown in [3] by cutting apart K_n and pasting copies together to form an infinite universal covering complex $\widetilde{K}_n$. The infinite complex is then realized as a packing $\widetilde{P}_n$ in $\mathbb{D}$ with symmetries corresponding to the action of $\pi_1(R_0)$.

It follows from the Length-Area Lemma [29] that the mesh of $\widetilde{P}_n$ decreases to 0 as n increases. Moreover, since the laminations inducing each E_m are formed from

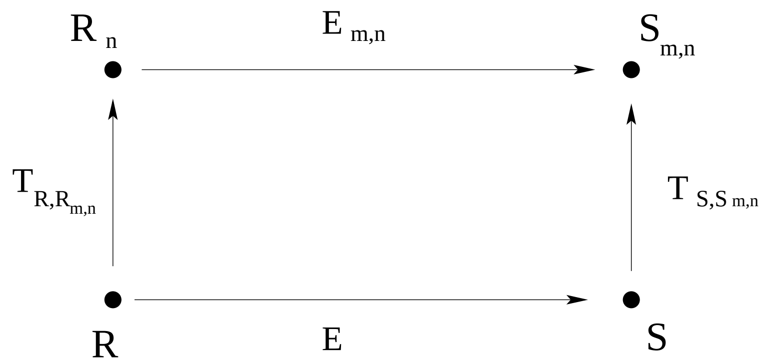

FIGURE 7. The action of the earthquakes and Teichmüller maps down on the surfaces.

finitely many simple closed curves, we can repeat the proof of Lemma 5.1 to form a combinatorial lamination corresponding to $\widetilde{\mathcal{L}}_{m,n}$.

For each fixed m and all large n, we combinatorially shear $\widetilde{P}_n$ along $\widetilde{\mathcal{L}}_{m,n}$. The resulting infinite complexes have packings $\widetilde{Q}_{m,n}$ filling $\mathbb{D}$. As before, we will normalize so that the endpoints of some geodesic are fixed.

The combinatorial earthquake produces a discrete earthquake map

$$\widetilde{E}_{m,n} : \operatorname{carr} \widetilde{P}_{m,n} \to \operatorname{carr} \widetilde{Q}_{m,n}.$$

Notice, however, that one difficulty in the proof of Theorem 1 does not occur here — the carriers of $\widetilde{P}_{m,n}$ and $\widetilde{Q}_{m,n}$ both completely fill $\mathbb{D}$ for each m and n, not simply in the limit.

Now observe that since $\widetilde{P}_n$ and $\widetilde{\mathcal{L}}_{m,n}$ are lifts of P_n and $\mathcal{L}_{m,n}$, respectively, the same packing and lamination are repeated on each fundamental region of R_n in $\mathbb{D}$. Hence the same combinatorial shearing operations must occur on each of these fundamental regions. As a result, $\widetilde{Q}_{m,n}$ must exhibit symmetries corresponding to the action of $\pi_1(R_n)$. In particular, this implies $\widetilde{Q}_{m,n}$ projects to a packing $Q_{m,n}$ on some hyperbolic Riemann surface $S_{m,n}$.

Moreover, since $\mathcal{L}_{m,n}$ was comprised of finitely many closed curves, each fundamental region of R_n in $\mathbb{D}$ intersects only finitely many strata of $\widetilde{\mathcal{L}}_{m,n}$. Thus we can define and extend our stratum maps as in Section 5.4.

Consequently, the same normality and dilatation arguments used in the proof of Theorem 1 imply the convergence of the stratum maps to hyperbolic isometries. Hence

$$\widetilde{E}_{m,n} \longrightarrow \widetilde{E}_m$$

as $n \to \infty$.

But recall that $E_m \to E$ as $m \to \infty$. Thus $\widetilde{E}_m$ converges to the lift $\widetilde{E}$ of E as $m \to \infty$. By a standard diagonalization argument, we may choose $n = n(m)$ for each m so that

$$\widetilde{E}_{m,n} \longrightarrow \widetilde{E}$$

as $m \to \infty$.

Now by projecting down to the surfaces, we see

$$(T_{S,S_{m,n}})^{-1} \circ E_{m,n} \circ T_{R,R_n} : R \to S \tag{6.1}$$

converges to E on R. See Figure 7. Since T_{R,R_n} converges to a conformal map of R which is homotopic to the identity, equation 6.1 implies $T_{S,S_{m,n}} = E_{m,n}T_{R,R_n}E^{-1}$ converges to a conformal map of S which is homotopic to the identity. But this is precisely the statement that the packable surfaces $S_{m,n}$ converge to S in the Teichmüller metric. Since S and R_0 were arbitrary, the theorem is proved. □

Notice that not every triangulation is obtainable from a given triangulation by earthquakes and hex refinement. (For example, fix a triangulation and barycentrically subdivide one of the faces.) Thus our earthquake deformations generate a proper dense subset of packable surfaces.

Moreover, following the proof of Theorem 2.3, we obtain a recipe for combinatorially deforming any packable surface into one arbitrarily close to any other surface in Teichmüller space.

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Department of Mathematics, Texas Tech University, Lubbock, Texas 79409
E-mail address: `williams@math.ttu.edu`
URL: `www.math.ttu.edu/~williams`