4.5 1st Hospital's rule

Theorem 4.9 (1st Hospital's rule)

Let \( f \) and \( g \) be differentiable functions with \( g'(c) \neq 0 \) on an open interval containing \( c \) (except possibly at \( c \) itself). Suppose \( \lim_{x \to c} \frac{f(x)}{g(x)} \) produces an indeterminate form \( 0/0 \) or \( \infty/\infty \) and that,

\[
\lim_{x \to c} \frac{f'(x)}{g'(x)} = L
\]

where \( L \) is either a finite number, \( +\infty \) or \( -\infty \). Then,

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = L
\]

The theorem also applies to one-sided limits and to limits at infinity. (where \( x \to \infty \) and \( x \to -\infty \).)

1. \( C \) can be \( 0 \) or a finite number
2. \( C^+ \) right-side limit
3. \( C^- \) left-side limit
4. \( \pm \infty \) limits at infinity.

Example 4.5.1

\[ \lim_{x \to 0} \frac{\sin x}{x} \]

Let \( f(x) = \sin x \) and \( g(x) = x \)

\[ \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0} \quad \text{Indeterminate Form} \]

\[ \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cos x}{1} = \lim_{x \to 0} \cos x = \cos 0 = 1 \]

(1st Hospital's rule)
\[ \lim_{x \to 0} \frac{1 - \cos x}{\sec x} \]

Let: \( f(x) = 1 - \cos x \) and \( g(x) = \sec x \)

Notice that \( \lim_{x \to 0} \frac{1 - \cos x}{\sec x} = \frac{1 - \cos 0}{\sec 0} = \frac{1-1}{1} = 0 \) and

\[ \text{this is not an indeterminate form.} \]

Suppose if you accidentally apply l'Hôpital's rule,

\[ \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\sin x}{\sec x \tan x} = \lim_{x \to 0} \frac{\sin x}{\frac{\cos x}{\sec x} \cdot \frac{\sin x}{\cos x}} \]

\[ = \lim_{x \to 0} \frac{\cos x}{\sec x} \]

\[ = \frac{\cos 0}{\sec 0} = \frac{1}{1} = 1 \quad \text{"wrong answer"} \]

Therefore, please remember to apply l'Hôpital's rule only when the limit is indeterminate.

\[ \therefore \]
\[ \lim_{x \to \infty} \frac{2x^3 + 3x + 1}{3x^2 + 5x - 2} \]

**Method 1**

\[ \lim_{x \to \infty} \frac{\frac{1}{x^2} (2x^3 + 3x + 1)}{\frac{1}{x^2} (3x^2 + 5x - 2)} = \lim_{x \to \infty} \frac{2 + \frac{3}{x} + \frac{1}{x^2}}{3 + \frac{5}{x} - \frac{2}{x^2}} = \frac{2}{3} \]

**Method 2**

\[ \lim_{x \to \infty} \frac{2x^2 + 3x + 1}{3x^2 + 5x - 2} = \frac{\infty}{\infty} \]

Apply 1' Hopital's rule:

\[ \lim_{x \to \infty} \frac{2x^2 + 3x + 1}{3x^2 + 5x - 2} = \lim_{x \to \infty} \frac{4x + 3}{6x + 5} = \frac{\infty}{\infty} \]

Apply 1' Hopital's rule again:

\[ \lim_{x \to \infty} \frac{2x^2 + 3x + 1}{3x^2 + 5x - 2} = \lim_{x \to \infty} \frac{4x + 3}{6x + 5} = \lim_{x \to \infty} \frac{4}{6} = \frac{4}{6} = \frac{2}{3} \]
\[ y = e^{-2x} \]

\[
\lim_{x \to \infty} x e^{-2x} = \lim_{x \to \infty} \frac{x}{e^{2x}} = \frac{\infty}{\infty} \quad \text{Undeterminate form.}
\]

Using l'Hôpital's rule,

\[
\lim_{x \to \infty} \frac{x}{e^{2x}} = \lim_{x \to \infty} \frac{1}{2e^{2x}} = 0
\]

\[
\lim_{x \to \infty} x e^{-2x} = \infty
\]

\[
\lim_{x \to \infty} \frac{2x^2}{2e^{2x}} = 0
\]

Therefore, \( y = 0 \) is a horizontal asymptote.
Other indeterminate forms

\[ \infty, \frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \frac{\infty}{\infty} \]

Convert into one of the standard form:

\[ \frac{0}{0}, \infty, \frac{\infty}{\infty} \]

Then evaluate using L'Hopital's rule.

Example 4.5.2

1. \( \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \rightarrow \infty \) indeterminate form

\[ \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \infty \] Indeterminate form.

\[ L = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \]

\[ \ln L = \ln \left( \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \right) \quad \text{Take logarithm of both sides} \]

\[ = \lim_{x \to \infty} \left[ \ln \left(1 + \frac{1}{x}\right)^x \right] \quad \text{Limit of a log property} \]

\[ = \lim_{x \to \infty} \left[ \infty \ln \left(1 + \frac{1}{x}\right) \right] \]

\[ = \lim_{x \to \infty} \left[ \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x^2}} \right] \quad \text{Write in terms of } 0/0 \]

\[ = \lim_{x \to \infty} \left\{ \frac{\left(\frac{1}{x^2}\right)(-\frac{1}{x^2})}{\left(1+\frac{1}{x^2}\right)} \right\} \quad \text{L'Hopital's rule} \]
\[
\begin{align*}
\lim_{x \to \infty} \left( \frac{1}{1 + \frac{1}{x}} \right) &= \frac{1}{1 + 0} \\
&= 1
\end{align*}
\]

\[
\ln L = 1
\]

\[
L = e
\]

\[
L = \lim_{x \to 0^+} x^{\sin x} \quad \text{indeterminate form}
\]

\[
\ln L = \ln \left( \lim_{x \to 0^+} x^{\sin x} \right)
\]

\[
= \lim_{x \to 0^+} \left( \ln x^{\sin x} \right)
\]

\[
= \lim_{x \to 0^+} \left( \sin x \ln x \right)
\]

\[
= \lim_{x \to 0^+} \frac{\ln x}{\cos x} \quad \text{this is of } \frac{\infty}{\infty} \text{ form.}
\]

\[
= \lim_{x \to 0^+} \frac{\ln x}{\csc x \cos x}
\]

\[
= \lim_{x \to 0^+} \frac{1}{\csc x \cot x}
\]

\[
= \lim_{x \to 0^+} \frac{1}{\csc x \cot x}
\]

\[
= \lim_{x \to 0^+} \frac{-\cos x / \sin^2 x}{x \cos x}
\]
\[
\lim_{x \to 0^+} \left( \frac{\sin x}{x} \right) \left( \frac{-\sin x}{\cos x} \right) = \left( \lim_{x \to 0^+} \frac{\sin x}{x} \right) \left( \lim_{x \to 0^+} \frac{-\sin x}{\cos x} \right) = 0
\]

\[\ln L = 0\]
\[L = e^0 = 1\]

\[L = \lim_{x \to \infty} x^{1/2} \quad \text{\(\infty/\infty\) indeterminate form.}\]

\[\ln L = \ln \left[ \lim_{x \to \infty} x^{1/2} \right] = \lim_{x \to \infty} \ln x^{1/2} = \lim_{x \to \infty} \frac{1}{2} \ln x = \lim_{x \to \infty} \frac{x}{1} \quad \text{1 Use Hospital's rule.}\]

\[= \frac{0}{1} = 0\]

\[L = e^0 = 1\]
Special limits involving $e^x$ and $\ln x$

**Theorem 4.10 (Limits involving natural logarithms and exponentials)**

If $k$ and $n$ are positive numbers,

\[
\lim_{x \to 0^+} \frac{\ln x}{x^n} = -\infty
\]

\[
\lim_{x \to \infty} \frac{\ln x}{x^n} = 0
\]

\[
\lim_{x \to \infty} \frac{e^{kx}}{x^n} = \infty
\]

\[
\lim_{x \to \infty} \frac{x^n}{e^{kx}} = 0
\]