5340-5341 Functional Analysis Fall 2001 -Spring 2002

1 Introduction.

1.1 Complete metric spaces. Theorem about completion.
1.2 Normed vector spaces. Banach spaces.
1.3 Examples of Banach spaces.

a) Spaces of sequences: $l^p (1 \leq p < \infty), l^\infty, c, c_0$.
b) Function spaces: $L^p(\Omega)(1 \leq p < \infty), L^\infty(\Omega), (\Omega \subset \mathbf{R^n}$ - open connected subset), $C(\bar{\Omega}), C_0(\bar{\Omega})(\Omega \subset \mathbf{R^n}$ is bounded domain).
c) Examples of infinite dimensional normed spaces that are not complete.

1.4 Separable spaces. Examples:

a) spaces $l^p, L^p(\Omega)(1 \leq p < \infty), c, C(\bar{\Omega})$ are separable;
b) spaces $l^\infty, L^\infty(\Omega)$ are not separable.

1.5 Equivalent norms. Subordination of norms.

a) Definition. Examples.
b) Equivalence of norms on finite dimensional spaces.
c) Completeness of finite dimensional normed spaces.
d) Examples of nonequivalent norms: stronger norms and weaker norms.

2 Hilbert spaces.

2.1 Inner product spaces.

a) Cauchy-Schwartz and triangle inequalities.
b) Parallelogram law and polarization identities.

2.2 Definition and examples of Hilbert spaces: l^2 and $L^2(\Omega)$.
2.3 Orthogonal projection theorem by F. Riesz.
2.4 Orthonormal systems.

a) Gram-Schmidt orthogonalization.
b) Bessel's inequality.
c) Projection of an arbitrary vector on a finite dimensional subspace.
d) Projection of an arbitrary vector on the closed linear span of an infinite orthonormal system.

2.5 Orthonormal bases.

a) Equivalent necessary and sufficient conditions for an orthonormal system to be an orthonormal basis:

(i) the orthogonal complement of the span is $\{0\}$;
(ii) the span is dense in the space;
(iii) Parceval's equality.

b) A Hilbert space is separable if and only if it has an orthonormal basis.

2.6 Examples of orthonormal bases

a) Standard basis in l^2.
b) Legendre polynomials.
c) Fourier series.

3 Linear bounded operators.

3.1 Definition. Norm of an operator.
3.2 Equivalence of boundedness and continuity.
3.3 Examples:

a) infinite matricies;
b) integral operators.

3.4 Extension by continuity.
3.5 Space $L(X, Y)$ of bounded linear operators (X and Y are normed vector spaces). Uniform convergence.
3.6 $L(X, Y)$ is a Banach space if Y is a Banach space.
3.7 Strong convergence and strong topology in $L(X, Y)$.

a) Example: Strong convergence does not imply uniform convergence.
b) Strong convergence of the shift operators to identity in $L^p(\Omega)(1 \leq p < \infty)$.

3.8 The uniform boundedness principle (Banach-Steinhaus theorem). Completeness of the space $L(X, Y)$ in strong topology if X and Y are Banach spaces.

4 Linear functionals and dual spaces.

4.1 Definitions.
4.2 Dual space of normed vector space is a Banach space.
4.3 The Hahn-Banach theorem.

a) Statement. Proof for separable normed vector spaces.
b) Corollaries.

4.4 Examples of dual spaces.

a) Dual space of $l^p(1 \leq p < \infty)$.
b) Dual space of $L^p(\Omega)(1 \leq p < \infty)$.

c) Dual space of $C(\bar{\Omega})$.
d) The Riesz representation theorem for linear functionals on a Hilbert space. Dual space of a Hilbert space.

4.5 Second dual space.

a) Isometric embedding of a space into its second dual.
b) Reflexive and nonreflexive Banach spaces. Examples.

4.6 Weak convergence of functionals. Weak completeness of dual spaces.
4.7 Weak convergence of vectors. Weak completeness of reflexive Banach spaces.
4.8 Weak continuity of bounded linear operators.
4.9 Weak convergence in specific spaces.

a) Weak convergence in a Hilbert space. Orthonormal systems in a Hilbert space weakly converge to zero.
b) Weak convergence in $l^p(1 \leq p < \infty)$.
c) Weak convergence in $L^p(\Omega)(1 \leq p < \infty)$.
d) Weak convergence in $C(\bar{\Omega})$.

5 Compact sets in Banach and Hilbert spaces.

5.1 Compact and relatively compact sets in metric spaces. ε-nets and totally bounded sets.
5.2 Normed vector space is locally compact if an only if it is finite dimensional.

a) Local compactness of finite dimensional normed spaces as a corollary of two facts:

(i) Heine-Borel theorem;
(ii) equivalence of norms on finite dimensional spaces.

b) Example: an infinite dimensional inner product space is not locally compact because an infinite orthonormal system cannot contain a convergent subsequence.
c) Riesz's lemma about almost orthogonal elements in a normed vector spaces.
d) Infinite dimensional normed spaces are not locally compact.

5.3 Compactness criteria in specific spaces.

a) Compact sets in a Hilbert space.
b) Compact sets in $C(\bar{\Omega})$. Askoli-Arzela's theorem.
c) Compact sets in $L^p(\Omega)$.

(i) Mollifiers and mollified functions.

(ii) Approximation of L^p-functions by their mollifications.
(iii) The case of a bounded domain Ω.
(iv) The case of an unbounded domain Ω.

5.4 Weak compactness.

a) Local weak compactness of the dual spaces of normed vector spaces: closed bounded sets in the dual space are weakly compact.
b) Local weak compactness of reflexive Banach spaces: closed bounded sets in a reflexive Banach space are weakly compact.
c) Banach space is reflexive if and only if it is weakly locally compact.

6 Compact operators

6.1 Definition and examples.

a) Integral operators.

(i) Integral operators in $L^2(\Omega)$ satisfying the Hilbert-Schmidt condition. Proof of compactness based on the compactness criterion in $L^2(\Omega)$.
(ii) Generalization to $L^p(\Omega)$ spaces ($1 \leq p < \infty$).

b) Infinite matricies.

(i) Matrix operators in l^2.
(ii) Generalization to l^p spaces ($1 \leq p < \infty$).

6.2 a) Uniform limit of compact operators is compact. The set of compact operators on a Banach space X form a closed two-sided ideal in the Banach algebra $L(X)$.
b) Strong limit of compact operators is not necessarily compact.

6.3 Alternative definition of compact operators in terms of weakly convergent sequences.

a) If $A \in L(X, Y)$ is compact then it maps weakly convergent sequences into strongly convergent sequences.
b) If $A \in L(X, Y)$ maps weakly convergent sequences into strongly convergent sequences and X is reflexive then A is compact.

6.4 Operators of finite rank on Hilbert spaces.

a) Definition.
b) General form of a finite rank operator.
c) Eigenvalues and eigenvectors. Reduction to a spectral problem from linear algebra.

d) Linear operator equations of the form $\lambda x - Ax = y$, where A is a finite rank operator.
e) Example: integral operators with degenerate kernels and degenerate integral equations.
f) Approximation of compact operators by operators of finite rank.
g) Alternative proof of compactness of a Hilbert-Schmidt integral operator.

7 Spectral theory of compact self-adjoint operators on a Hilbert space

7.1 Adjoint operators.

a) Definition.
b) Norm of the adjoint operator
c) Orthogonal sum decomposition of the space in terms of the null spaces and ranges of an operator and its adjoint.
d) Remark: the kernel of a bounded operator is always closed while the range may be not closed. Examples.
e) Adjoint of an integral operator in $L^2(\Omega)$ and of a matrix operator in l^2.

7.2 Self-adjoint bounded operators.

a) Definition and examples.
   a) Integral operators with Hermitian Kernels.
   b) Infinite Hermitian matricies.
b) An operator is self-adjoint if and only if its quadratic form is real valued.
c) Orthogonal projectors. A bounded operator is an orthogonal projector if and only if it is idempotent and self-adjoint.
d) Norm of a self-adjoint operator is equal to the supremum of the absolute value of its quadratic form on the unit sphere.

7.3 The problem of existence of eigenvalues and eigenvectors.

a) Two counterexamples.
   (i) Bounded self-adjoint operator which has no eigenvalues.
   (ii) Compact nonself-adjoint operator which has no eigenvalues.
b) If an operator A is compact and self-adjoint then it has an eigenvalue, whose absolute value is equal to the norm of A.

7.4 Spectral theorem for compact self-adjoint operators.

a) Statement of the theorem.

   (i) Existence of a basic system of eigenvectors and eigenvalues: an orthonormal system (finite or infinite) of eigenvectors such that the corresponding eigenvalues are not equal to zero but converge to zero if the system is infinite.
   (ii) Spectral decomposition in terms of the basic system of eigenvalues and eigenvectors.

b) Proof of the theorem.

7.5 Comments on the spectral theorem.

a) Point $\lambda = 0$ may or may not be an eigenvalue. Its multiplicity may be either finite or infinite.
b) There are no eigenvalues except for the basic system and possibly zero.
c) Nonzero eigenvalues have finite multiplicities.
d) Nonuniqueness of eigenvectors if the multiplicities of the eigenvalues are greater than one.
e) If the number of nonzero eigenvalues is infinite then the range of the operator is never closed.
f) Basic system of eigenvectors forms an orthonormal basis in the closure of the range of the operator.
g) The entire Hilbert space has an orthonormal basis consisting of a basic system of eigenvectors plus eigenvectors corresponding to possible eigenvalue zero.
h) Converse of the spectral theorem: every operator that admits the spectral decomposition with real eigenvalues is compact and self-adjoint.
i) Statement of the spectral decomposition in terms of spectral projectors (orthogonal projectors on eigenspaces).

7.6 Resolvent of a compact self-adjoint operator

a) Linear operator equations of the form $\lambda x - Ax = y$, where A is a compact self-adjoint operator.

   (i) The case when $\lambda \neq 0$ and is not an eigenvalue.
   (ii) The case when λ is a nonzero eigenvalue.
   (iii) The case when $\lambda = 0$.

b) Formula for the resolvent in terms of a basic system of eigenvectors and eigenvalues. The resolvent is an analytic meromorphic function of the spectral parameter. Its poles are the eigenvalues and the corresponding residues are the spectral projectors.

7.6 The mini-max theorem.

8 Applications of the spectral theory of compact self-adjoint operators.

8.1 Self-adjoint integral operators.

- a) Spectral theorem for integral operators in $L^2(\Omega)$ with Hermitian kernels satisfying the Hilbert-Schmidt condition.
- b) The Hilbert-Schmidt theorem. Absolute and uniform convergence of the spectral decomposition.

8.2 Self-adjoint Sturm-Liouville operators.

- a) Definition
  - (i) Definition of a formally self-adjoint differential Sturm-Liouville operator.
  - (ii) Linear first order boundary conditions at the ends of a finite interval.
  - (iii) Conditions on the coefficients. Regularity of the operator.
  - (iv) Sturm-Liouville operator as an unbounded operator in the space $L^2(a, b)$. The domain of the operator.
- b) Statement of the spectral theorem.
- c) The Green's function.
- d) Reduction to an integral equation and proof of the spectral theorem.

9 More applications of the spectral theorem

9.1 Spectral theorem for compact normal operators.

- a) Simultaneous diagonalization of commuting operators.
- b) The real and imaginary part of a bounded operator on a complex Hilbert space.
- c) Two equivalent definitons of a normal operator:
  - (i) in terms of an operator and its adjoint;
  - (ii) in terms of the real and imaginary part.
- d) Statement and proof of the spectral theorem.

9.2 Functions of compact self-adjoint and normal operators.

9.3 Unitary equivalence of compact self-adjoint operators.

- a) Unitary operators.
- b) Necessary and sufficient condition for unitary equivalence.

9.4 General form of a compact nonself-adjoint operator on a Hilbert space.

10 Fredholm-Riesz-Schauder theory of compact operators on Banach spaces

10.1 Adjoint operators.
10.2 Statement of Fredholm theorems. Fredholm alternative.
10.3 Spectrum of a compact operator.
10.4 Approximation by finite rank operators in Banach spaces with basis.
10.5 Proof of Fredholm theorems based on finite rank approximation.
10.6 Riesz-Schauder proof of Fredholm theorems.

11 Sobolev spaces

11.1 Introduction.

- a) Multi-index notations for derivatives.
- b) Classical spaces of continuous and differentiable functions: $C^l(\bar{\Omega}); l = 0, 1, \ldots; \Omega \subset \mathbb{R}^n$- subdomain.
- c) Review of basic facts about spaces $L^p(\Omega)(1 \leq p \leq \infty)$.
- d) Definition of Sobolev norms.
- e) Nonconstructive definition of Sobolev spaces $W_p^l(\Omega)$ as completions of spaces $C^l(\bar{\Omega})$ in Sobolev norms $(1 \leq p < \infty)$.

11.2 Mollifiers and mollified functions.

- a) Mollifiers or averaging kernels.
- b) Basic properties of mollified functions.
- c) Approximation by mollified functions.
- d) $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)(1 \leq p < \infty)$.
- e) Space $L^1_{loc}(\Omega)$ and the generalized main lemma of the calculus of variations.

11.3 Weak derivatives.

- a) Multidimensional integration by parts formula.
- b) Definition of weak derivatives.
- c) Discussion.
  - (i) Uniqueness.
  - (ii) Linearity.
  - (iii) Weak derivative operators commute with mollification operators on strictly inner subdomains.
  - (iv) An equivalent definition of weak derivatives.
- d) Properties of weak derivatives.
  - (i) Product rule
  - (ii) Chain rule.

(iii) If all weak derivatives up to the order l of a given function are equal to zero then the function is a polynomial of degree $l-1$.
(iv) Weakly differentiable functions of one variable.
(v) First weak derivatives in dimension n.
(vi) The existence of higher order weak derivative does not imply the existence of lower order weak derivatives.

11.4 Definition and basic properties of Sobolev spaces.

a) Definition of the spaces $W_p^l(\Omega), 1 \le p \le \infty, l = 0, 1, \ldots,$ in terms of weak derivatives.
b) $W_p^l(\Omega)$ is a Banach space.
c) $W_p^l(\Omega)$ is separable and reflexive if $1 \le p < \infty$.
d) Space $\overset{o}{W}{}_p^l(\Omega)$: closure of $C_0^\infty(\Omega)$ in $W_p^l(\Omega)$-norm.
e) Hilbert Sobolev spaces $H_p^l(\Omega)$ and $H_0^l(\Omega)$.
f) $C^l(\Omega)$ is dense in $W_p^l(\Omega)$ (proof for star-like domains): return to the original definition.

11.5 Extension theorem.

11.6 Statements of embedding theorems:

a) General concept of an embedding. Examples based on spaces of sequences.
b) Sobolev embedding theorem.
c) Compact embeddings: Rellich-Kondrashov theorem.
d) The trace operator: conditions for boundedness and compactness.

11.7 Proofs of some embedding theorems.

a) Equivalent description of spaces $H_0^l(Q)$ (Q is a cube in $\mathbb{R}^n$) in terms of Fourier series.
b) Proof of embedding theorems for spaces $H^l(\Omega)$ based on the extension theorem and Fourier series representation.
c) Sobolev integral representation for compactly supported smooth functions on a ball.
d) Integral operators with polar kernels. Conditions for boundedness and compactness.
e) Proofs of embedding theorems based on extension theorem, Sobolev integral representation, and the results about operators with polar kernels.

12 Elliptic differential operators.

12.1 Second order elliptic boundary value problem.