Numerical Analysis II Spr 2010

1. Ackleh *et. al.* Ch 6: 11, 12, 13, 19, 22

2. An important building block in finite element methods is the approximation of integrals over a triangle T by quadrature,
$$\int_T f(x,y)\,dx\,dx \approx Q(f) = \sum_{i=1}^{N} w_i f(x_i, y_i).$$
Work on the unit equilateral triangle (*i.e.* the equilateral triangle that fits exactly inside the unit circle), and let $C^k(\overline{B})$ be the set of functions that are k-times differentiable on the closure of the unit ball, $\overline{B} = \{(x,y) \mid x^2 + y^2 \le 1\}$.

   (a) Find the point and weight for a one-point quadrature rule that can integrate exactly all linear functions
   $$f_1(x,y) = a_{00} + a_{10}x + a_{01}y$$
   on the unit equilateral triangle. Find an upper bound on the error when this rule is used to approximate the integral of an arbitrary function $f \in C^2(\overline{B})$.

   (b) Find points and weights for a three-point quadrature rule that can integrate exactly all quadratic functions
   $$f_2(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{02}y^2 + a_{11}xy$$
   on the unit equilateral triangle. Find an upper bound on the error when this rule is used to approximate the integral of an arbitrary function $f \in C^3(\overline{B})$.

3. Use the Golub-Welsch algorithm and the recurrence relation for the Hermite orthogonal polynomials to derive an eigenvalue problem with which you can compute nodes and weights for Gauss-Hermite quadrature with $N = 6$. Do not confuse the Hermite polynomials with the Hermite *interpolating* polynomials. Check your results against a verified table of Gauss-Hermite nodes and weights (available online, or in reference books such as Abramowitz and Stegun's *Handbook of Mathematical Functions*.)

4. Approximate the integrals
$$I_1 = \int_0^{2\pi} e^{2\cos x}\,dx$$
and
$$I_2 = \int_0^{10} e^{2\cos x}\,dx$$
using the following methods: N-point CTR and N-point Gauss-Legendre, for $N = 2, 4, 8, 16, 32$. Plot the errors on a log-log scale. Can you explain the results?

5. Approximate the integrals
$$I_1 = \int_0^{1} \sqrt{x}\,dx$$
and
$$I_2 = \int_1^{2} \sqrt{x}\,dx$$
using the following methods: N-point Gauss-Legendre for $N = 4, 8, 16, 32$, and composite 4-point Gauss-Legendre with $N = 1, 2, 4, 8$ subdivisions. Plot the errors on a log-log scale. Can you explain the results?