Homework 5 solutions

October 25, 2009

Lecture 8

Problem 8.2

See the code file mgs.m. Here is output on a small test matrix:

```
A = rand(8,5)

A =

    0.3517    0.3804    0.5688    0.1656    0.2290
    0.8308    0.5678    0.4694    0.6020    0.9133
    0.5853    0.0759    0.0119    0.2630    0.1524
    0.5497    0.0540    0.3371    0.6541    0.8258
    0.9172    0.5308    0.1622    0.6892    0.5383
    0.2858    0.7792    0.7943    0.7482    0.9961
    0.7572    0.9340    0.3112    0.4505    0.0782
    0.7537    0.1299    0.5285    0.0838    0.4427


[Q1,R1] = mgs(A)

Q1 =

    0.1874    0.1714    0.4150   -0.3134   -0.4634
    0.4426    0.0535    0.0286    0.0437    0.7232
    0.3118   -0.3065   -0.1877    0.0522   -0.2532
    0.2929   -0.3062    0.2746    0.5968   -0.3211
    0.4887   -0.0430   -0.3824    0.2504   -0.0333
    0.1523    0.6382    0.4633    0.3527    0.0673
    0.4034    0.4909   -0.4156   -0.3030   -0.2676
    0.4016   -0.3605    0.4276   -0.5119    0.1336


R1 =

    1.8770    1.1691    0.9548    1.2372    1.3606
         0    0.9419    0.4779    0.4185    0.2799
         0         0    0.7424    0.1478    0.7317
         0         0         0    0.6356    0.7049
         0         0         0         0    0.3380
```

```
% compare to reduced QR from MATLAB:
[Q2,R2] = qr(A,0)

Q2 =

   -0.1874    0.1714    0.4150    0.3134    0.4634
   -0.4426    0.0535    0.0286   -0.0437   -0.7232
   -0.3118   -0.3065   -0.1877   -0.0522    0.2532
   -0.2929   -0.3062    0.2746   -0.5968    0.3211
   -0.4887   -0.0430   -0.3824   -0.2504    0.0333
   -0.1523    0.6382    0.4633   -0.3527   -0.0673
   -0.4034    0.4909   -0.4156    0.3030    0.2676
   -0.4016   -0.3605    0.4276    0.5119   -0.1336


R2 =

   -1.8770   -1.1691   -0.9548   -1.2372   -1.3606
         0    0.9419    0.4779    0.4185    0.2799
         0         0    0.7424    0.1478    0.7317
         0         0         0   -0.6356   -0.7049
         0         0         0         0   -0.3380
```

Note that the methods agree up to signs.

Lecture 10

Problem 10.2

See the codes house.m and formQ.m. Here is output on the same test matrix as in 8.2.

```
[W,R3] = house(A)

W =

    0.7705         0         0         0         0
    0.2872   -0.7108         0         0         0
    0.2023   -0.2472   -0.7915         0         0
    0.1901   -0.2451    0.1363    0.9034         0
    0.3171   -0.0798   -0.3405    0.1813    0.8265
    0.0988    0.4335    0.2104    0.2832    0.4482
    0.2618    0.3044   -0.3858   -0.0919    0.2597
    0.2606   -0.2943    0.2144   -0.2496   -0.2204


R3 =

   -1.8770   -1.1691   -0.9548   -1.2372   -1.3606
```

```
   -0.0000    0.9419    0.4779    0.4185    0.2799
   -0.0000         0    0.7424    0.1478    0.7317
   -0.0000         0   -0.0000   -0.6356   -0.7049
   -0.0000         0    0.0000    0.0000   -0.3380
   -0.0000         0   -0.0000    0.0000    0.0000
   -0.0000         0    0.0000   -0.0000    0.0000
   -0.0000   -0.0000   -0.0000   -0.0000   -0.0000


Q3 = formQ(W)

Q3 =

  Columns 1 through 6

   -0.1874    0.1714    0.4150    0.3134    0.4634   -0.2052
   -0.4426    0.0535    0.0286   -0.0437   -0.7232    0.0494
   -0.3118   -0.3065   -0.1877   -0.0522    0.2532    0.7953
   -0.2929   -0.3062    0.2746   -0.5968    0.3211   -0.1590
   -0.4887   -0.0430   -0.3824   -0.2504    0.0333   -0.4673
   -0.1523    0.6382    0.4633   -0.3527   -0.0673    0.2769
   -0.4034    0.4909   -0.4156    0.3030    0.2676    0.0375
   -0.4016   -0.3605    0.4276    0.5119   -0.1336   -0.0343

  Columns 7 through 8

   -0.2877   -0.5705
    0.1216   -0.5082
   -0.2495   -0.1095
    0.5067   -0.0615
   -0.5397    0.1992
   -0.1992    0.3311
    0.4995    0.0960
    0.0549    0.4919

% Compare with full QR from MATLAB
Q4 =

  Columns 1 through 6

   -0.1874    0.1714    0.4150    0.3134    0.4634   -0.2052
   -0.4426    0.0535    0.0286   -0.0437   -0.7232    0.0494
   -0.3118   -0.3065   -0.1877   -0.0522    0.2532    0.7953
   -0.2929   -0.3062    0.2746   -0.5968    0.3211   -0.1590
   -0.4887   -0.0430   -0.3824   -0.2504    0.0333   -0.4673
   -0.1523    0.6382    0.4633   -0.3527   -0.0673    0.2769
   -0.4034    0.4909   -0.4156    0.3030    0.2676    0.0375
   -0.4016   -0.3605    0.4276    0.5119   -0.1336   -0.0343

  Columns 7 through 8
```

```
  -0.2877   -0.5705
   0.1216   -0.5082
  -0.2495   -0.1095
   0.5067   -0.0615
  -0.5397    0.1992
  -0.1992    0.3311
   0.4995    0.0960
   0.0549    0.4919


R4 =

  -1.8770   -1.1691   -0.9548   -1.2372   -1.3606
        0    0.9419    0.4779    0.4185    0.2799
        0         0    0.7424    0.1478    0.7317
        0         0         0   -0.6356   -0.7049
        0         0         0         0   -0.3380
        0         0         0         0         0
        0         0         0         0         0
        0         0         0         0         0
```

Problem 10.4

(a) Left multiplication by J rotates the plane $\mathbb{R}^2$ by angle θ in the clockwise direction. For example, if $\theta = \pi/2$, the vector $x = [1; 0]$ is rotated 90^o clockwise to $[0; -1]$:

$$\begin{aligned} J &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ Jx &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{aligned}$$

Left multiplication by F is a reflection. If the vector x is at an angle α from the positive x-axis, it is reflected to the line at angle $\pi - \theta - \alpha$.

(b) and **(c)** coming soon...

Lecture 11

Problem 11.1

Claim: $\|A^+\|_2 \leq \|A_1^{-1}\|_2$

Proof Let $P = A(A^*A)^{-1}A^*$ be an orthogonal projector onto range(A) and let $x \in \mathbb{C}^n$ be nonzero. We can split x in to its component in the range of A and a remainder: $x = w + r$, where $w = Px \in$ range(A) and $r \in$ null(A^*). Then we have (assuming 2-norms throughout):

$$\begin{aligned}
\|A^+\| &\leq \frac{\|Ax\|}{\|x\|} \\
&= \frac{\|A^+(w+r)\|}{\|x\|} \\
&= \frac{\|(A^*A)^{-1}A^*(w+r)\|}{\|w+r\|} \\
&= \frac{\|(A^*A)^{-1}A^*w\|}{\|w+r\|} \quad \text{since } r \in \text{null}(A^*) \\
&= \frac{\|(A^*A)^{-1}A^*(Ay)\|}{\|(Ay)+r\|} \quad \text{for some } y \text{ (since } w \in \text{ range}(A)) \\
&= \frac{\|y\|}{\|Ay+r\|} \\
&\leq \frac{\|y\|}{\|Ay\|} \\
&\leq \frac{\|y\|}{\|A_1 y\|} \\
&\leq \|A_1^{-1}\|
\end{aligned}$$

∎

Problem 11.3

```
m = 50;
n = 12;
t = linspace(0,1,m);
t = t';
A = vander(t);
A = fliplr(A);
A = A(:,1:n);
b = cos(4*t);
format long

% (a) Solution via normal equations:
B = A'*A;
bb = A'*b;
```

```
xnormal = B\bb;
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 2.992867e-17.

% (b) Solution via modified Gram-Schmidt with mgs.m
[Q1,R1] = mgs(A);
y = Q1'*b;
xmgs = R1\y;

% (c) Solution via Householder with house.m and formQ.m
[W,R2] = house(A);
Q2 = formQ(W);
y = Q2'*b;
xhouse = R2\y;

% (d) Solution via matlab's qr
[Q3,R3] = qr(A);
y = Q3'*b;
xqr = R3\y;

% (e) Solution via matlab's backslash
xbslash = A \ b;

% (f) Solution via SVD
[U,S,V] = svd(A);
y = U'*b;
w = S \ y;
xsvd = V*w;
```

(g)

xnormal	xmgs	xhouse	xqr	xbslash	xsvd
1.000000007972887	0.999999998220264	1.000000000996601	1.000000000996609	1.000000000996608	1.000000000996608
-0.000002500274451	0.000000379645838	-0.000000422743032	-0.000000422742968	-0.000000422743228	-0.000000422742965
-7.999902739915186	-8.000011127015082	-7.999981235683213	-7.999981235687960	-7.999981235679865	-7.999981235688125
-0.001481595176351	0.000118974101235	-0.000318763277182	-0.000318763203417	-0.000318763301264	-0.000318763200829
10.678376546894064	10.666096445657857	10.669430796252895	10.669430795705138	10.669430796332966	10.669430795684868
-0.054370245025213	0.001163511539917	-0.013820289530095	-0.013820287197629	-0.013820289609699	-0.013820287102762
-5.531642469895595	-5.689404847687352	-5.647075623245154	-5.647075629418464	-5.647075623539531	-5.647075629701436
-0.287488484503984	0.001960170466892	-0.075316031327473	-0.075316020815308	-0.075316030120297	-0.075316020265539
1.945133753544229	1.602552674723292	1.693606971184400	1.693606959644216	1.693606969166215	1.693606958952933
-0.179680484459608	0.072894805908085	0.006032103477705	0.006032111376332	0.006032105309351	0.006032111917519
-0.296574794164493	-0.402066671802883	-0.374241701390361	-0.374241704457822	-0.374241702274479	-0.374241704697128
0.073989380164090	0.093052068359714	0.088040575723150	0.088040576239507	0.088040575901462	0.088040576285090

The normal equations are the least reliable. Multiplying A by A^* squares the condition number. (Note that we got a warning about nearly singular matrices during the normal equations solve. QR via Household, MATLAB's QR, and solutions via the SVD all appear to be stable. QR via MGS looks less so.