Homework 4 solutions
October 21, 2009

Lecture 6

Problem 6.4

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(a)

$$\begin{aligned}
P &= A(A^*A)^{-1}A^* \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \\
Py &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}
\end{aligned}$$

(b)

$$
\begin{aligned}
P &= B(B^*B)^{-1}B^* \\
&= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \\
Py &= \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}
\end{aligned}
$$

Problem 6.5

Let $P \in \mathbb{C}^{m \times m}$ be a nonzero projector.
Claim $\|P\|_2 \geq 1$

Proof Let $x \in \text{range}(P)$. Then $Px = x$ and $\|Px\|_2 = \|x\|_2$. Thus $\|P\|_2 = \sup_{\|x\|=1} \|Px\|_2 \geq 1$.
■

Claim $\|P\|_2 = 1$ if and only if P is an orthogonal projector.

Proof If P is an orthogonal projector, then $P = Q\Sigma Q^*$ where

$$\Sigma = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

(See proof of Theorem 6.1 in NLA.) Therefore $\|P\|_2 = \sigma_1 = 1$.
Now we'll assume $\|P\|_2 = 1$. Let $u \in \text{range}(P)$ and $v \in \text{null}(P) = \text{range}(I - P)$ be nonzero

vectors. Decompose u orthogonally into its component in the direction of v and the remainder r.

$$\begin{aligned} u &= \frac{vv^*}{v^*v}u + r \\ &= v\frac{v^*u}{v^*v} + r \\ Pu &= Pv\frac{v^*u}{v^*v} + Pr \\ &= Pr \ \ (\text{since } v \in \ \text{range(P)}) \\ &= u \ \ (\text{since } u \in \text{range(P)}) \end{aligned}$$

Therefore we have

$$\begin{aligned} \|Pr\|_2^2 &= \|u\|_2^2 \\ &= \left(\frac{v^*u}{v^*v}v + r\right)^* \left(\frac{v^*u}{v^*v}v + r\right) \\ &= \left(\frac{v^*u}{v^*v}\right)^2 v^*v + r^*r \\ &= \left(\frac{v^*u}{v^*v}\right)^2 \|v\|_2^2 + \|r\|_2^2 \end{aligned}$$

At the same time, we know that

$$\begin{aligned} \|Pr\|_2^2 &\leq \|P\|_2^2\|r\|_2^2 \\ &= \|r\|_2^2 \ \ (\text{since } \|P\|_2 = 1) \end{aligned}$$

Therefore, v^*u must be 0. The vectors u and v were arbitrary nonzero members of range(P) and null(P), and we have shown that $v \perp u$. So the two spaces must be orthogonal and P is an orthogonal projects. ∎

Lecture 7

Problem 7.1

(a) Since working on paper, for the reduced $\hat{Q}\hat{R}$ we could use Gram-Schmidt or modified Gram-Schmidt. To get a full QR, we can add arbitrary ON columns to $\hat{Q}$ using Gram-Schmidt and zero rows to $\hat{R}$. Or,we could use Householder to get a full QR directly.

$$\begin{aligned} A &= \hat{Q}\hat{R} \\ &= \begin{bmatrix} -1/\sqrt{2} & 0 \\ 0 & 1 \\ -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
A &= QR \\
&= \begin{bmatrix} -1/\sqrt{2} & 0 & -\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

(b)

$$
\begin{aligned}
B &= \hat{Q}\hat{R} \\
&= \begin{bmatrix} -1/\sqrt{2} & -\sqrt{3}/3 \\ 0 & -\sqrt{3}/3 \\ -1/\sqrt{2} & \sqrt{3}/3 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & -\sqrt{2} \\ 0 & -\sqrt{3} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
B &= QR \\
&= \begin{bmatrix} -1/\sqrt{2} & -\sqrt{3}/3 & -1/\sqrt{6} \\ 0 & -\sqrt{3}/3 & 2/\sqrt{6} \\ -1/\sqrt{2} & \sqrt{3}/3 & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -\sqrt{2} & -\sqrt{2} \\ 0 & -\sqrt{3} \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

Problem 7.5

(a) We have A $= \hat{Q}\hat{R}$, so the columns of A can be expressed as

$$
\begin{aligned}
a_1 &= r_{11}q_1 \\
a_2 &= r_{12}q_1 + r_{22}q_2 \\
&\vdots \\
a_n &= r_{1n}q_1 + r_{2n} + cdots + r_nq_2
\end{aligned}
$$

If $r_{jj} \neq 0\ \forall j$, then each column a_j of A is a linear combination of q_1 through q_j and each column includes a q_j that was not included in the previous columns. Therefore, they columns of A are linearly independent, i.e., A has full rank.

If $r_{jj} = 0$ for some j, then $a_j = r_{1j}q_1 + r_{2j}q_2 + \cdots + r_{j-1,j}q_{j-1} + 0q_j$ is a linear combination of q_1 through q_{j-1} and therefore of a_1 through a_{j-1}. Thus A is not full rank.

(b) If $\hat{R}$ has k nonzero diagonal entries, then again using

$$
\begin{aligned}
a_1 &= r_{11}q_1 \\
a_2 &= r_{12}q_1 + r_{22}q_2 \\
&\vdots \\
a_n &= r_{1n}q_1 + r_{2n} + cdots + r_nq_2
\end{aligned}
$$

we see that the columns of A are made up of linear combinations of at least k linearly independent q_j columns. Therefore, the rank of A is at least k.

Now we need to see if it is exactly k or if it can be greater than k. In fact, it can be greater than k. Consider the matrix:

$$\hat{R} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 1 \\ & & 0 & 1 & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

This $\hat{R}$ has 1 nonzero diagonal entry, but rank $4 > 1$. Therefore, if $\hat{R}$ has k nonzero diagonals, A has rank *at least* k.

Lecture 8

Problem 8.1

```
1:  for i = 1 to n do
2:      v_1 = a_1
3:  end for
4:  for i = 1 to n do
5:      r_ii = ||v_i||
6:      q_1 = v_i / r_ii
7:      for j = i + 1 to n do
8:          r_ij = q_i^* v_j
9:          v_j = v_j - r_ij q_i
10:     end for
11: end for
```

In the outer loop (lines 5 and 6), we have m multiplications, $m-1$ additions, and one square root for $\|v_i\|$, and m divisions for v_i/r_{ii}, for a total of $3m$ flops. These operations are executed n times.

In the inner loop (lines 8 and 9), we have m multiplications and $m-1$ additions for the dot product $q_i^* v_j$, m multiplications for the scalar times vector $r_{ij}q_i$, and m subtractions for the vector-vector subtraction $v_j - r_{ij}q_j$, for a total of 4m-1 flops. These operations are executed $n-i$ times for each outer loop $i = 1$ to n.

Therefore, the total exact flop count is:

$$\begin{aligned}
\text{flops} &= \sum_{i=1}^{n} 3m + \sum_{i=1}^{n} \sum_{j=i+1}^{n} 4m - 1 \\
&= 3mn + \sum_{i=1}^{n} (4m-1)(n-i) \\
&= 3mn + \sum_{i=1}^{n} 4mn - n - 4mi + i \\
&= 3mn + 4mn^2 - n^2 - 2mn(n+1) + n(n+1)/2 \\
&= 2mn^2 + mn - \frac{1}{2}n^2 + \frac{1}{2}n
\end{aligned}$$

Note that if we leave out the square root in the calculation of the norm in line 5 (I would count it, but it wasn't explicitly mentioned in the problem), we would get:

$$\begin{aligned}
\text{flops} &= \sum_{i=1}^{n} 3m - 1 + \sum_{i=1}^{n} \sum_{j=i+1}^{n} 4m - 1 \\
&= 2mn^2 + mn - \frac{1}{2}n^2 - \frac{1}{2}n
\end{aligned}$$