Homework 2 solutions

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Lecture 3

Problem 3.1

We want to show that $\|x\|_W = \|Wx\|$ is a norm when $\|\cdot\|$ is a norm. So we just need to show that $\|\cdot\|_W$ obeys the three vector norm properties (3.1):

1. $\|x\|_W \geq 0$, and $\|x\|_W = 0$ only if $x = 0$
2. $\|x+y\|_W \leq \|x\|_W + \|y\|_W$
3. $\|\alpha x\|_W = |\alpha|\|x\|_W$

We'll take them in order:

1. $\|x\|_W = \|Wx\| \geq 0$ since Wx is a vector and $\|\cdot\|$ is a norm. Since W is assumed to be nonsingular (and therefore $null(W) = \{0\}$), $Wx = 0$ only if $x = 0$. Therefore, $\|x\|_W = \|Wx\| = 0$ only if $Wx = 0$ (since $\|\cdot\|$ is a norm), only if $x = 0$ (since W is nonsingular).

2. $$\begin{aligned} \|x+y\|_W &= \|W(x+y)\| \\ &= \|Wx + Wy\| \\ &\leq \|Wx\| + \|Wy\| \quad \text{(since } \|\cdot\| \text{ is a norm)} \\ &= \|x\|_W + \|y\|_W \end{aligned}$$

3. $$\begin{aligned} \|\alpha x\|_W &= \|W\alpha x\| \\ &= \|\alpha W x\| \\ &= |\alpha|\|Wx\| \quad \text{since } \|\cdot\| \text{ is a norm)} \\ &= |\alpha|\|x\|_W \end{aligned}$$

Problem 3.3

(a)

$$\begin{aligned}
\|x\|_2 &= \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} \\
&= \left(|x_1|^2 + |x_2|^2 + \cdots + |x_m|^2\right)^{1/2} \\
&\geq \left(\max_{1\leq i\leq m} |x_i|^2\right)^{1/2} \\
&\quad \text{(since each term is non-negative, the sum is the max term plus some non-negative number)} \\
&= \max_{1\leq i\leq m} |x_i| \\
&= \|x\|_\infty
\end{aligned}$$

For an example where equality is achieved, consider any constant c times a column of the identity matrix ce_j. $\|ce_j\|_2 = |c|$ and $\|ce_j\|_\infty = |c|$.

(b)

$$\begin{aligned}
\|x\|_2 &= \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} \\
&= \left(|x_1|^2 + |x_2|^2 + \cdots + |x_m|^2\right)^{1/2} \\
&\leq \left(\max_{1\leq i\leq m} |x_i|^2 + \max_{1\leq i\leq m} |x_i|^2 + \cdots + \max_{1\leq i\leq m} |x_i|^2\right)^{1/2} \\
&= \left(m \max_{1\leq i\leq m} |x_i|^2\right)^{1/2} \\
&= \sqrt{m}\|x\|_\infty
\end{aligned}$$

For an example where equality is achieved, consider the vector of all ones $x = (1, 1, \ldots, 1)^*$. Then $\|x\|_2 = \sqrt{m}$ and $\|x\|_\infty = 1$.

(c) First note that for $A \in \mathbb{C}^{m\times n}$ that domain elements $x \in \mathbb{C}^n$ and range elements $Ax \in \mathbb{C}^m$.

$$\begin{aligned}
\frac{\|Ax\|_\infty}{\|x\|_\infty} &\leq \frac{\|Ax\|_2}{\|x\|_\infty} && \text{for any x by part (a)} \\
\frac{\|Ax\|_\infty}{\sqrt{n}\|x\|_\infty} &\leq \frac{\|Ax\|_2}{\|x\|_2} && \text{for any x by part (b)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\|Ax\|_\infty}{\|x\|_\infty} &\leq \sqrt{n}\frac{\|Ax\|_2}{\|x\|_2} \quad \text{for any vector x} \\
\Rightarrow \sup_{x\neq 0}\frac{\|Ax\|_\infty}{\|x\|_\infty} &\leq \sup_{x\neq 0}\sqrt{n}\frac{\|Ax\|_2}{\|x\|_2} \\
\therefore \ \|A\|_\infty &\leq \sqrt{n}\|A\|_2
\end{aligned}$$

For an example where equality is achieved, consider a matrix with ones in the first row and zeros elsewhere. Then $\|A\|_2 = \sqrt{n}$ and $\|A\|_\infty = n$.

(d)

$$\begin{aligned}
\frac{\|Ax\|_2}{\|x\|_2} &\leq \sqrt{m}\frac{\|Ax\|_\infty}{\|x\|_2} \quad \text{for any x by part (b)} \\
\Rightarrow \frac{\|Ax\|_2}{\|x\|_2} &\leq \sqrt{m}\frac{\|Ax\|_\infty}{\|x\|_\infty} \quad \text{for any x by part (a)} \\
\Rightarrow \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} &\leq \sup_{x\neq 0}\sqrt{m}\frac{\|Ax\|_\infty}{\|x\|_\infty} \\
\therefore \ \|A\|_2 &\leq \sqrt{m}\|A\|_\infty
\end{aligned}$$

For an example where equality is achieved, consider a matrix with ones in the first column and zeros elsewhere. Then $\|A\|_2 = \sqrt{m}$ and $\|A\|_\infty = 1$.

Problem 3.5

If E is an outer product $E = uv^*$, then the equation $\|E\|_F = \|u\|_F\|v\|_F$ is true for the Frobenius norm.

Proof Let e_j be the j^{th} column of $E \in \mathbb{C}^{m\times n}$ where E is an outer product $E = uv^*$. Then $e_j = v_j u$ (see the outer product in Example 1.2 of NLA).

$$\begin{aligned}
\|E\|_F &= \left(\sum_{j=1}^{n} \|e_j\|_2^2\right)^{1/2} \quad \text{(from (3.17) in NLA)} \\
&= \left(\sum_{j=1}^{n} \|v_j u\|_2^2\right)^{1/2} \\
&= \left(\sum_{j=1}^{n} |v_j|^2 \|u\|_2^2\right)^{1/2} \\
&= \left(\|u\|_2^2 \sum_{j=1}^{n} |v_j|^2\right)^{1/2} \\
&= \|u\|_2 \left(\sum_{j=1}^{n} |v_j|^2\right)^{1/2} \\
&= \|u\|_2\|v\|_2
\end{aligned}$$

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Problem 3.6

(a) We need to show that the three norm conditions hold for $\|\cdot\|'$.

1. $\|x\|' = \sup_{\|y\|=1} |y^*x| \geq 0 \quad \forall\ x, y$. If $x \neq 0$ then $\|x\| \neq 0$ (since $\|\cdot\|$ is a norm). Let $y = x/\|x\|$ (so $\|y\| = 1$). Then

$$\begin{aligned}
|y^*x| &= |x^*x/\|x\|| &> 0 \text{ for this y} \\
\Rightarrow \sup_{\|y\|=1} &\geq |x^*x/\|x\|| &> 0 \\
\Rightarrow \|x\|' &\geq |x^*x/\|x\|| &> 0
\end{aligned}$$

   Therefore, $\|x\|' \geq 0$, and $\|x\|' = 0$ only if $x = 0$.

2.

$$\begin{array}{rcl}
|y^*(x_1+x_2)| & = & |y^*x_1 + y^*x_2| \\
 & \leq & |y^*x_1| + |y^*x_2| \\
\text{Therefore, } \sup\limits_{\|y\|=1} |y^*(x_1+x_2)| & \leq & \sup\limits_{\|y\|=1} |y^*x_1| + \sup\limits_{\|y\|=1} |y^*x_2| \\
\therefore\ \ \|x_1+x_2\|' & \leq & \|x_1\|' + \|x_2\|'
\end{array}$$

3.

$$\begin{array}{rcl}
|y^*(\alpha x)| & = & |\alpha y^* x| \\
 & = & |\alpha||y^*x| \ \text{ for any x, y} \\
\Rightarrow \sup\limits_{\|y\|=1} |y^*(\alpha x)| & = & \sup\limits_{\|y\|=1} |\alpha||y^*x| \\
 & = & |\alpha| \sup\limits_{\|y\|=1} |y^*x| \\
\therefore\ \ \|\alpha x\|' & = & |\alpha|\|x\|'
\end{array}$$

(b)

Proof Let $x, y \in \mathbb{C}^m$ with $\|x\| = \|y\| = 1$ be given. Let $\hat{z}$ be the z that exists from the lemma, i.e., $|\hat{z}^*x| = \|\hat{z}\|'\|x\|$. Now let $\tilde{z} = \hat{z}/\|\hat{z}\|'$. Then $|\tilde{z}^*x| = \|\tilde{z}\|'\|x\| = \|x\| = 1$. Since $|\tilde{z}^*x| = 1$, $\tilde{z}^*x$ is equal to 1 times a scalar of modulus 1. Therefore there is a scalar α such that $\alpha\tilde{z}^*x = 1$ (without the absolute values). Let $z = \alpha\tilde{z}$. Then $z^*x = 1$.

Let $B = yz^*$ with this z. Then B is rank 1, and $Bx = yz^*x = y$.

Now we just need to show that $\|B\| = 1$ in the matrix norm induced by the vector norm $\|\cdot\|$.

$$\begin{array}{rcl}
\|B\| & = & \sup\limits_{\|w\|=1} \|Bw\| \\
 & = & \sup\limits_{\|w\|=1} \|yz^*w\| \\
 & = & \sup\limits_{\|w\|=1} \|y(z^*w)\| \\
 & = & \sup\limits_{\|w\|=1} |z^*w|\|y\| \\
 & = & \sup\limits_{\|w\|=1} |z^*w| \\
 & = & \sup\limits_{\|w\|=1} |w^*z| \\
 & = & \|z\|' \\
 & = & 1
\end{array}$$

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