
Homework 1 solutions

September 15, 2009

Lecture 1

Problem 1.1

(a)

$$\begin{array}{ccccccc}
 \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & [B] & \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 (5) & (3) & (2) & & (1) & (4) & (6) & (7)
 \end{array}$$

(b)

$$ABC = \begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} [B] \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 1.4

(a) This problem represents interpolation of 8 data points d_i by a linear combination of 8 basis functions f_i . Let

$$d = \begin{bmatrix} d_1 \\ \vdots \\ d_8 \end{bmatrix} \in \mathbb{C}^8, \quad c = \begin{bmatrix} c_1 \\ \vdots \\ c_8 \end{bmatrix} \in \mathbb{C}^8, \quad \text{and} \quad F = \begin{bmatrix} f_1(1) & f_2(1) & \cdots & f_8(1) \\ f_1(2) & f_2(2) & \cdots & f_8(2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(8) & f_2(8) & \cdots & f_8(8) \end{bmatrix} \in \mathbb{C}^{8 \times 8}$$

Since for any $d \in \mathbb{C}^8$ $\exists c \in \mathbb{C}^8$ such that $d = Fc$, the mapping is onto and the range of F is \mathbb{C}^8 . By Theorem 1.3, F has full rank and F^{-1} exists. Since $c = F^{-1}d$, c is determined by d . Since F^{-1} must also have full rank (since it is invertible), F^{-1} (and F) map no two vectors to the same vector (Theorem 1.2), i.e., they are one-to-one mappings; thus d determines c uniquely.

(b) A is the 8×8 matrix mapping d to the coefficients c , i.e., $Ad = c$. So $A = F^{-1}$ from part (a). $A^{-1} = (F^{-1})^{-1} = F$. Therefore,

$$(A^{-1})_{ij} = (F)_{ij} = f_j(i).$$

Lecture 2

Problem 2.1

Let $A \in \mathbb{C}^{m \times m}$ be triangular and unitary. Assume without loss of generality (“wlog”) that A is upper triangular. Then if a_{ij} is the element of A at row i and column j , then

$$a_{ij} = 0 \text{ for } i > j.$$

Since A is invertible (since it’s unitary), A^{-1} exists, and by Problem 1.3 A^{-1} is also upper triangular. So if \tilde{a}_{ij} is the i, j^{th} element of A^{-1} ,

$$\tilde{a}_{ij} = 0 \text{ } i > j.$$

Since A is unitary, we have

$$A^* = A^{-1}$$

Problem 2.2

(a) For $n = 2$,

$$\begin{aligned} \|\sum_{i=1}^2 x_i\|_2^2 &= \|(x_1 + x_2)\|_2^2 \\ &= (x_1 + x_2)^*(x_1 + x_2) \\ &= x_1^*x_1 + x_1^*x_2 + x_2^*x_1 + x_2^*x_2 \\ &= x_1^*x_1 + x_2^*x_2 \\ &= \|x_1\|_2^2 + \|x_2\|_2^2 \\ &= \sum_{i=1}^2 \|x_i\|_2^2 \end{aligned}$$

(b) Now assume that the proposition is true for n and show that it’s true for $n + 1$:

$$\begin{aligned} \|\sum_{i=1}^{n+1} x_i\|_2^2 &= \|(x_1 + \cdots + x_{n+1})\|_2^2 \\ &= (x_1 + \cdots + x_{n+1})^*(x_1 + \cdots + x_{n+1}) \\ &= ((x_1 + \cdots + x_n) + x_{n+1})^* ((x_1 + \cdots + x_n) + x_{n+1}) \\ &= (x_1 + \cdots + x_n)^*(x_1 + \cdots + x_n) + x_{n+1}^*(x_1 + \cdots + x_n) \\ &\quad + (x_1 + \cdots + x_n)^*x_{n+1} + x_{n+1}^*x_{n+1} \\ &= \sum_{i=1}^n \|x_i\|_2^2 + \|x_{n+1}\|_2^2 \\ &= \sum_{i=1}^{n+1} \|x_i\|_2^2 \quad \square \end{aligned}$$

Problem 2.6

Let u and v be m -vectors and let $A = I + uv^*$.

Let α be given by

$$\alpha = -\frac{1}{1 + v^*u}$$

Then,

$$\begin{aligned}(I + uv^*)(I + \alpha uv^*) &= I + uv^* + \alpha uv^* + \alpha uv^* uv^* \\ &= I + uv^* + \alpha uv^* + \alpha u(v^*u)v^* \\ &= I + uv^* + \alpha(1 + v^*u)uv^* \\ &= I + uv^* - \left(\frac{1}{1 + v^*u}\right)(1 + v^*u)uv^* \\ &= I\end{aligned}$$

Similarly for $(I + \alpha uv^*)(I + uv^*)$. The inverse is unique if it exists. So if there is an inverse, it has this form.

If $A = (I + uv^*)$ is singular, then there is an $x \neq 0$ such that $(I + uv^*)x = 0$. I.e.,

$$\begin{aligned}(I + uv^*)x &= 0 \\ x + uv^*x &= 0 \\ uv^*x &= -x\end{aligned}$$

In other words, -1 is an eigenvalue of uv^* and x is the corresponding eigenvector. Thus $v^*u = -1$ since uv^* and v^*u have the same nonzero eigenvalues. This is also the value of v^*u for which α does not exist.

The null space of $A = (I + uv^*)$ is the space spanned by $\mathbf{0}$ and the x that is a solution to $(I + uv^*)x = 0$.