Math 1352-11 — WW09 Solutions

November 24, 2008

Assigned problems: 8.7 – 10, 16, ww_4; 8.8 – 32, ww_5, ww_6

Always read through the solution sets even if your answer was correct. Note that like many of the integrals in this course, there is frequently more than one way to determine convergence or divergence of a series. Your solution may be correct even if you used a different method than what I use here.

1. (8.7 #10) $\sum_{k=1}^{\infty} \frac{(x-15)^k}{\ln(k+1)}$: First we'll notice that the series converges (trivially) for $x = 15$ since each term is identically 0. For $x \neq 15$, we'll try the **generalized ratio test**:

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(x-15)^k \;/\; \ln(k+2)}{(x-15)^k \;/\; \ln(k+1)}\right| \\
&= \lim_{k\to\infty}\left|\frac{\ln(k+1)(x-15)^{k+1}}{\ln(k+2)(x-15)^k}\right| \\
&= \lim_{k\to\infty}\frac{\ln(k+1)}{\ln(k+2)}\,|x-15| \\
&= \lim_{k\to\infty}\frac{(k+2)}{(k+1)}\,|x-15| \quad \textit{(L'hopital)} \\
&= \lim_{k\to\infty}\cancelto{1}{\frac{(k+2)}{(k+1)}}\,|x-15| \\
&= |x-15| \;=\; L
\end{aligned}$$

According to the generalized ratio test, the series converges for $L < 1$ and diverges for $L > 1$. So this series converges when $|x = 15| < 1$, i.e., when $-1 < (x - 15) < 1$ which gives us $14 < x < 16$.

Similar, the series converges when $L > 1$, so $|x - 15| > 1$ giving $x > 16$ or $x < 14$.

The generalized ratio test is inconclusive when $L = |x - 15| = 1$, so we need to check the endpoints $x = 14$ and $x = 16$ separately.

When $x = 14$, we have

$$\sum_{k=1}^{\infty}\frac{(x-15)^k}{\ln(k+1)} \;=\; \sum_{k=1}^{\infty}\frac{(-1)^k}{\ln(k+1)}$$

This is an alternating series, so we'll test with the alternating series test.

$$\lim_{k\to\infty}\frac{1}{\ln(k+1)} = 0$$

$$\begin{aligned}
\ln(k+2) &> \ln(k+1) \\
\frac{1}{\ln(k+2)} &< \frac{1}{\ln(k+1)} \\
a_{k+1} &< a_k
\end{aligned}$$

Therefore the series converges at $x = 14$.

When $x = 16$,

$$\sum_{k=1}^{\infty} \frac{(x-15)^k}{\ln(k+1)} = \sum_{k=1}^{\infty} \frac{1}{\ln(k+1)}$$

We'll do a limit comparison test with $\sum b_k = \sum \frac{1}{k}$:

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{1/\ln(k+1)}{1/k} \\
&= \lim_{k\to\infty} \frac{k}{\ln(k+1)} \\
&= \lim_{k\to\infty} \frac{1}{1\ (k+1)} \quad \textit{(L'hopital)} \\
&= \lim_{k\to\infty} \frac{(k+1)}{1} = \infty
\end{aligned}$$

This limit goes to infinity and $\sum b_k$ diverges, so by the $0-\infty$ **limit comparison test** the series diverges, too.

So the interval of convergence is $\boxed{14 \le x < 16}$.

2. (8.7 #16) $\sum_{k=1}^{\infty} \frac{(2k)!x^k}{(3k)!}$:

First we'll notice that the series converges (trivially) for $x = 0$ since each term is identically 0. For $x \neq 0$, we'll try the **generalized ratio test**:

$$\begin{aligned}
\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|\frac{(2k+2)!x^{k+1}/(3k+3!}{(2k)!x^k/(3k)!}\right| \\
&= \lim_{k\to\infty} \left|\frac{(3k)!(2k+2)!x^{k+1}}{(3k+3)!(2k)!x^k}\right| \\
&\quad \textit{(Note } (2k+2)! = (2k+2)(2k+1)(2k)! \textit{ and } (3k+3)! = (3k+3)(3k+2)(3k+1)(3k)!\textit{)} \\
&= \frac{(2k+2)(2k+1)}{(3k+3)(3k+2)(3k+1)}|x| \\
&= \frac{4k^2+6k+2}{27k^3+45k^2+21k+3}|x| \\
&= \frac{4/k+6/k^2+2/k^3}{27+45/k+21/k^2+3/k^3}|x| \\
&= 0
\end{aligned}$$

The limit is 0 for any value of x. Therefore the $\boxed{\textbf{series converges absolutely for any } x, -\infty < x < \infty}$. (Since the interval is the whole real line, there are no endpoints to check.)

3. (8.8 #32) First 4 coefficients of Taylor series for $f(x) = \ln(x)$ at $c = 3$. We'll need the first 3 derivatives of f evaluated at $c = 3$:

$$\begin{array}{rclrcl} f(x) & = & \ln(x) & f(3) & = & \ln(3) \\ f'(x) & = & \frac{1}{x} & f'(3) & = & \frac{1}{3} \\ f''(x) & = & \frac{-1}{x^2} & f''(3) & = & -\frac{1}{9} \\ f'''(x) & = & \frac{2}{x^3} & f'''(3) & = & \frac{2}{27} \end{array}$$

So the Taylor series at $c = 3$ for $f(x) = \ln(x)$ is:

$$\begin{aligned} f(x) &= f(3) + \frac{f'(3)(x-3)}{1} + \frac{f''(3)(x-3)^2}{2!} + \frac{f'''(c)(x-3)^3}{3!} + \cdots \\ &= \ln(3) + (\frac{1}{3})(x-3) + \frac{(-1/9)(x-3)^2}{2} + \frac{(2/27)(x-3)^3}{6} + \cdots \\ &= \ln(3) + \frac{(x-3)}{3} - \frac{(x-3)^2}{18} + \frac{(x-3)^3}{81} + \cdots \\ &= c_0 + c_1(x-3) + c_2(x-3)^2 + c_3(x-3)^3 + \cdots \end{aligned}$$

where

$$\begin{aligned} c_0 &= \ln(3) \\ c_1 &= 1/3 \\ c_2 &= -1/18 \\ c_3 &= 1/81 \end{aligned}$$

4. (Note that webwork creates random, but similar, problems for each student on this question. So the solutions here are for a sample of problems to give you some examples. If you have questions on any particular problem you were given, please ask.

(4.1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+3}$:
This is an alternating series, so we can try the **alternating series test**. We need to check that $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \le a_k$.

$$\begin{aligned} \lim_{k\to\infty} a_k &= \lim_{k\to\infty} \frac{1}{4n+3} \\ &= 0 \end{aligned}$$

And,

$$\begin{aligned} 4n+7 &> 4n+3 \\ 4(n+1)+3 &> 4n+3 \\ \frac{1}{4(n+1)+3} &< \frac{1}{4n+3} \\ a_{n+1} &< a_n \end{aligned}$$

So the the series converges.

Now we'll check if it converges absolutely.

$$\sum \left| \frac{(-1)^n}{4n+3} \right| \quad = \quad \sum \frac{1}{4n+3}$$

This looks almost like a divergent p-series. We can confirm that it diverges with a **limit comparison test** with $\sum b_k = \sum \frac{1}{k}$.

$$\begin{aligned} \lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{1/(4k+3}{1/k} \\ &= \lim_{k\to\infty} \frac{k}{4k+3} \\ &= \frac{1}{4} \end{aligned}$$

The limit is positive and finite, so the two series converge or diverge together. Therefore $\sum |a_k|$ diverges. So the original series $\boxed{\textbf{converges conditionally}}$.

(4.2) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+2}$:
This is an alternating series, so we can try the **alternating series test**. We need to check that $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$.

$$\begin{aligned} \lim_{k\to\infty} a_k &= \lim_{k\to\infty} \frac{\sqrt{k}}{k+2} \\ &= \lim_{k\to\infty} \frac{(1/2)k^{-1/2}}{1} \quad \textit{(L'hopital)} \\ &= \lim_{k\to\infty} \frac{1}{2\sqrt{n}} \\ &= 0 \end{aligned}$$

And,

$$\begin{aligned} a_{k+1} &\leq a_k \ \textit{?} \\ \frac{\sqrt{k+1}}{k+3} &\leq \frac{\sqrt{k}}{k+2} \ \textit{?} \\ \frac{\sqrt{k+1}(k+2)}{\sqrt{k}(k+3)} &\leq 1 \ \textit{?} \\ \frac{(k+1)(k+2)^2}{k(k+3)^2} &\leq 1 \ \textit{?} \\ \frac{k^3+5k^2+8k+4}{k^3+6k^2+9k} &\leq 1 \ \textit{?} \\ k^3+5k^2+8k+4 &\leq k^3+6k^2+9k \ \textit{?} \\ 4 &\leq k^2+k \ \textit{?} \end{aligned}$$

This is true for all $x \geq 2$. The conditions of the alternating series test are met, so the **series converges**.

Now we'll check if the series converges absolutely. Consider

$$\sum \left| \frac{(-1)^n \sqrt{n}}{n+2} \right| \quad = \quad \sum \frac{\sqrt{n}}{n+2}$$

We'll do a **direct comparison test** with $\sum b_k = \sum 1/(n+2)$ (which diverges by a limit comparison with $\sum 1/n$).

$$\frac{\sqrt{n}}{n+2} \geq \frac{1}{n+2} \geq 0$$

So the series is term by term $\geq$ a divergent series. So by the direct comparison test, the series diverges. Therefore, since the original series converges but the absolute value series diverges, the original series $\boxed{\textbf{converges conditionally}}$.

(4.3) $\sum_{n=1}^{\infty} \frac{(n+1)(6^2-1)^n}{6^{2n}}$:
Because of the powers of n, I'll try the **generalized root test**.

$$\begin{aligned}
\lim_{n\to\infty} |\sqrt[n]{a_n}| &= \lim_{n\to\infty} \left| \frac{(n+1)^{1/n}(6^2-1)}{(6^{2n})^{1/n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^{1/n}(6^2-1)}{6^2} \right| \\
&= \lim_{n\to\infty} \frac{35}{36}(n+1)^{1/n} \\
& \quad \textit{(This is } \infty^0 \textit{ form)} \\
L &= \lim_{n\to\infty} \frac{35}{36}(n+1)^{1/n} \\
\frac{36}{35}L &= \lim_{n\to\infty} (n+1)^{1/n} \\
\ln\left(\frac{36}{35}L\right) &= \ln\left(\lim_{n\to\infty} (n+1)^{1/n}\right) \\
\ln\left(\frac{36}{35}L\right) &= \lim_{n\to\infty} \ln\left((n+1)^{1/n}\right) \\
\ln\left(\frac{36}{35}L\right) &= \lim_{n\to\infty} \frac{1}{n}\ln(n+1) \\
\ln\left(\frac{36}{35}L\right) &= \lim_{n\to\infty} \frac{\ln(n+1)}{n} \\
\ln\left(\frac{36}{35}L\right) &= \lim_{n\to\infty} \frac{1}{n+1} \quad \textit{(L'hopital)} \\
\ln\left(\frac{36}{35}L\right) &= 0 \\
\frac{36}{35}L &= 1 \quad \textit{(Take exponential of both sides.)} \\
L &= \frac{35}{36}
\end{aligned}$$

The limit is < 1, therefore the series $\boxed{\textbf{converges absolutely}}$.

(4.4) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$:

We'll try the **generalized ratio test**.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-2)^{n+1}/(n+1)^2}{(-2)^n/n^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^2(-2)^{n+1}}{(n+1)^2(-2)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{2n^2}{(n+1)^2}\right| \\
&= \lim_{n\to\infty}\frac{2n^2}{n^2+2n+1} \\
&= \lim_{n\to\infty}\frac{2}{1+2/n+1/n^2} \\
&= 2
\end{aligned}$$

The limit is > 1, so the $\boxed{\textbf{series diverges}}$.

(4.5) $\sum_{n=1}^{\infty} \frac{\sin(5n)}{n^2}$:

This series can be negative, but does not alternate in a regular $+, -$ pattern, so it's not an alternating series.

I'll try testing for absolute convergence on

$$\sum_{n=1}^{\infty}\left|\frac{\sin(5n)}{n^2}\right|$$

We can use a direct comparison test. Since $|sin(x)| \leq 1$ for any x, we have

$$0 \leq \left|\frac{\sin(5n)}{n^2}\right| \leq \frac{1}{n^2}$$

Therefore, the absolute value series is dominated by the convergent p-series $\sum \frac{1}{n^2}$ (p = 2). Therefore the series is $\boxed{\textbf{absolutely convergent}}$.

5. Here we are matching a set of series with the Maclaurin series for different functions. I'll derive the Maclaurin series (Taylor series at $c = 0$) for the given functions. Each time, we'll get one of the given series.

- $f(x) = \cos(x)$

$$\begin{array}{rcrrcr}
f(x) &=& \cos(x) & f(0) &=& 1 \\
f'(x) &=& -\sin(x) & f'(0) &=& 0 \\
f''(x) &=& -\cos(x) & f''(0) &=& -1 \\
f'''(x) &=& \sin(x) & f'''(0) &=& 0 \\
f^{(4)}(x) &=& \cos(x) & f^{(4)}(0) &=& 1 \\
\ldots & & & & &
\end{array}$$

$$\begin{aligned}
\cos(x) &= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}
\end{aligned}$$

- $f(x) = e^x$

$$\begin{array}{rclrcl}
f(x) &=& e^x & f(0) &=& 1 \\
f'(x) &=& e^x & f'(0) &=& 1 \\
f''(x) &=& e^x & f''(0) &=& 1 \\
\cdots &&&&&
\end{array}$$

$$\begin{aligned}
e^x &= 1 + \frac{1 \cdot x}{1!} + \frac{1 \cdot x^2}{2!} + \frac{1 \cdot x^3}{3!} + \frac{1 \cdot x^4}{4!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!}
\end{aligned}$$

- $f(x) = \arctan(x)$

$$\begin{array}{rcl rcr}
f(x) &=& \arctan(x) & f(0) &=& 0 \\
f'(x) &=& \frac{1}{1+x^2} & f'(0) &=& 1 \\
f''(x) &=& -(2x)(1+x^2)^{-2} & f''(0) &=& 0 \\
f'''(x) &=& (8x^2)(1+x^2)^{-3} - 2(1+x^2)^{-2} & f'''(0) &=& -2 \\
f^{(4)}(x) &=& (16x)(1+x^2)^{-3} - (48x^3)(1+x^2)^{-4} + (8x)(1+x^2)^{-3} & f^{(4)}(0) &=& 0 \\
\cdots &&&&&
\end{array}$$

$$\begin{aligned}
\arctan(x) &= 0 + x + 0 - \frac{2x^3}{3!} + \cdots \\
&= x - \frac{x^3}{3} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}
\end{aligned}$$

- $f(x) = \sin(x)$

$$\begin{array}{rcr rcr}
f(x) &=& \sin(x) & f(0) &=& 0 \\
f'(x) &=& \cos(x) & f'(0) &=& 1 \\
f''(x) &=& -\sin(x) & f''(0) &=& 0 \\
f'''(x) &=& -\cos(x) & f'''(0) &=& -1 \\
f^{(4)}(x) &=& \sin(x) & f^{(4)}(0) &=& 0 \\
\cdots &&&&&
\end{array}$$

$$\begin{aligned}
\sin(x) &= 0 + \frac{1 \cdot x}{1!} + 0 - \frac{1 \cdot x^3}{3!} + 0 + \frac{1 \cdot x^5}{5!} + \cdots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}
\end{aligned}$$

6. Taylor series for $f(x) = \cos(x)$ at $c = \pi/2$.

$$\begin{array}{rcrrcr}
f(x) &=& \cos(x) & f(\pi/2) &=& 0 \\
f'(x) &=& -\sin(x) & f'(\pi/2) &=& -1 \\
f''(x) &=& -\cos(x) & f''(\pi/2) &=& 0 \\
f'''(x) &=& \sin(x) & f'''(\pi/2) &=& 1 \\
f^{(4)}(x) &=& \cos(x) & f^{(4)}(\pi/2) &=& 0 \\
f^{(5)}(x) &=& -\sin(x) & f^{(5)}(\pi/2) &=& 1 \\
\cdots & & & & &
\end{array}$$

$$\begin{aligned}
\cos(x) &= 0 - \frac{1 \cdot (x - \pi/2)}{1!} + 0 + \frac{1 \cdot (x - \pi/2)^3}{3!} + 0 - \frac{1 \cdot (x - \pi/2)^5}{5!} + \cdots \\
&= \sum_{k=0}^{\infty} c_k (x - \pi/2)^k
\end{aligned}$$

Where the first 5 coefficients are:

$$\begin{aligned}
c_0 &= 0 \\
c_1 &= -1 \\
c_2 &= 0 \\
c_3 &= 1/6 \\
c_4 &= 0
\end{aligned}$$