Math 1352-11 — WW07 Solutions

November 15, 2008

Assigned problems: 8.1 – 12, 16, 26, 8.2 – 4, 6, 12, 28, ww_8, 8.3 – 4, 12, 20, 36

Always read through the solution sets even if your answer was correct.

Note that like many of the integrals in this course, there is frequently more than one way to determine convergence or divergence of a series. Your solution may be correct even if you used a different method than what I use here.

1. (8.1 #12)

$$\lim_{n\to\infty} \frac{5n+8}{n} = \lim_{n\to\infty} \frac{5+8/n}{1} = 5$$

$\boxed{\textbf{Sequence converges to 5.}}$

2. (8.1 #16)

$$\begin{aligned}
\lim_{n\to\infty} \frac{8n^2+800n+5000}{2n^2-1000n+2} &= \lim_{n\to\infty} \frac{16n+800}{4n-1000} \quad \textit{(using L'hopital's rule)} \\
&= \lim_{n\to\infty} \frac{16}{4} \quad \textit{(L'hop. again)} \\
&= 4
\end{aligned}$$

$\boxed{\textbf{Sequence converges to 4.}}$

3. (8.1 #26)

$$\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n$$

This is similar to a limit you should remember from Calc I:

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

(If you do not remember this limit, review section 2.4 of the textbook. I will assume that you know this limit.) Here, we need to do a substitution to turn the given limit into one involving the limit we know. Let $u = n/3$. This means that $n = 3u$ and $3/n = 1/u$. Note also that $u \to \infty$ as $n \to \infty$. So we can rewrite our limit as:

$$\begin{aligned}
\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n &= \lim_{u\to\infty} \left(1+\frac{1}{u}\right)^{3u} \\
&= \lim_{u\to\infty} \left[\left(1+\frac{1}{u}\right)^u\right]^3 \\
&= \left[\lim_{u\to\infty} \left(1+\frac{1}{u}\right)^u\right]^3 = e^3
\end{aligned}$$

$\boxed{\textbf{Sequence converges to } e^3\textbf{.}}$

4. (8.2 #4)

$$\sum_{k=0}^{\infty} \left(-\frac{4}{5}\right)^k$$

Geometric series with $a = 1$ and $r = -4/5$. **Converges** since $|r| = 4/5 < 1$. Converges to:

$$\frac{a}{1-r} = \frac{1}{1+\frac{4}{5}} = \frac{5}{9}$$

$\boxed{\textbf{Series converges to 5/9.}}$

5. (8.2 #6)

$$\sum_{k=0}^{\infty} \frac{2}{(-3)^k} \quad = \quad \sum_{k=0}^{\infty} 2\left(-\frac{1}{3}\right)^k$$

Geometric series with $a = 2$ and $r = -\frac{1}{3}$. **Converges** since $|r| = 1/3 < 1$. Converges to:

$$\frac{a}{1-r} = \frac{2}{1+\frac{1}{3}} = \frac{6}{4} = \frac{3}{2}$$

$\boxed{\textbf{Series converges to 3/2.}}$

6. (8.2 #12)

$$\sum_{k=1}^{\infty} \frac{3^k}{4^{k+2}} = \sum_{k=1}^{\infty} \frac{3^k}{4^2 4^k} = \sum_{k=1}^{\infty} \left(\frac{1}{16}\right)\left(\frac{3}{4}\right)^k$$

Geometric series with $r = 3/4$. **Converges** since $|r| = 3/4 < 1$. Converges to:

$$\begin{aligned} \frac{a}{1-r} &= \frac{\frac{3}{4^3}}{1-\frac{3}{4}} \quad \textit{(Note that "a" is the first term in the series, which is when k = 1 in this case.)} \\ &= \frac{3}{64-48} \\ &= \frac{3}{16} \end{aligned}$$

$\boxed{\textbf{Series converges to 3/16.}}$

7. (8.2 #28)

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}$$

Note that the partial fraction hint in webwork for this problem seems to be wrong and leads you to the wrong answer. I will not count this problem in your ww07 grade. We do want to try partial fractions on this series however.

$$\begin{aligned}
\frac{1}{(2k-1)(k2+1)} &= \frac{A}{2k-1} + \frac{B}{2k+2} \\
1 &= A(2k+1) + B(2k-1) \\
1 &= 2Ak + A + 2Bk - B \\
1 &= (2A+2B)k + (A-B)
\end{aligned}$$

Therefore we know that $2A + 2B = 0$ and $A - B = 1$. So we find that $A = 1/2$ and $B = -1/2$. And,

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \sum_{k=1}^{\infty} \frac{1/2}{2k-1} - \frac{1/2}{2k+1}$$

We know that the sum of the series S is the limit of the partial sums S_n. So first, we'll see if we can find a general form for S_n.

$$\begin{aligned}
S_n &= \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{2n-3} - \frac{1}{2n-1} + \frac{1}{2n-1} - \frac{1}{2n+1}\right] \\
& \quad \textit{(Note that the series telescopes and we can cancel terms.)} \\
&= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right]
\end{aligned}$$

Now we can take the limit of the partial sums as $n \to \infty$:

$$\begin{aligned}
S &= \lim_{n\to\infty} S_n \\
&= \lim_{n\to\infty} \frac{1}{2}\left[1 - \frac{1}{2n+1}\right] \\
&= \frac{1}{2}
\end{aligned}$$

$\boxed{\textbf{The telescoping sequence converges to 1/2.}}$

8.

$$\sum_{n=1}^{\infty} [\ln(2(n+1)) - \ln(2n)]$$

I'll try the divergence test first. The sequence diverges if $\lim_{n\to\infty} a_n \neq 0$.

$$\begin{aligned}
\lim_{n\to\infty} [\ln(2(n+1)) - \ln(2n)] &= \lim_{n\to\infty} \ln\left(\frac{(2(n+1))}{(2n)}\right) \\
&= \ln\left(\lim_{n\to\infty} \frac{2n+2}{2n}\right) \\
&= \ln\left(\lim_{n\to\infty} \frac{2+2/n}{2}\right) \\
&= \ln(1) \\
&= 0
\end{aligned}$$

The limit is 0, so the divergence test is inconclusive (may converge or diverge). We'll try looking at the nth partial sum.

$$\begin{aligned}
S_n &= [\ln(4) - \ln(2)] + [\ln(6) - \ln(4)] + [\ln(8) - \ln(6)] + \cdots + [\ln(2n) - \ln(2n-2)] + [\ln(2n+2) - \ln(2n)] \\
&= [\cancel{\ln(4)} - \ln(2)] + [\cancel{\ln(6)} - \cancel{\ln(4)}] + [\cancel{\ln(8)} - \cancel{\ln(6)}] + \cdots + [\cancel{\ln(2n)} - \cancel{\ln(2n-2)}] + [\ln(2n+2) - \cancel{\ln(2n)}] \\
&= -\ln(2) + \ln(2n+2)
\end{aligned}$$

$$\begin{aligned}
S &= \lim_{n\to\infty} S_n \\
&= \lim_{n\to\infty} [-\ln(2) + \ln(2n+2)] \\
&= \infty
\end{aligned}$$

The limit goes to infinity, so $\boxed{\textbf{the series diverges}}$.

9. (8.3 #4)

$$\sum_{k=1}^{\infty} \frac{100}{\sqrt{k}} = \sum_{k=1}^{\infty} 100 \cdot \frac{1}{k^{1/2}}$$

$\boxed{\textbf{Divergent p-series since } p = \tfrac{1}{2} \leq 1.}$

10. (8.3 #12)

$$
\begin{aligned}
\int_1^\infty xe^{-x^2}\,dx &= \lim_{N\to\infty}\int_1^N xe^{-x^2}\,dx \\
&\quad \textit{(Let } u = x^2,\ du = 2x\,dx\textit{)} \\
&= \lim_{N\to\infty}\int_{x=1}^{x=N}\frac{1}{2}e^{-u}\,du \\
&= \lim_{N\to\infty} -\frac{1}{2}e^{-u}\Big|_{x=1}^{x=N} \\
&= \lim_{N\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^N \\
&= \lim_{N\to\infty}\left(-\frac{1}{2}e^{-N^2}+\frac{1}{2}e^{-1}\right) \\
&= \lim_{N\to\infty}\left(-\frac{1}{2e^{N^2}}+\frac{1}{2e}\right) \\
&= \frac{1}{2e}
\end{aligned}
$$

So the improper integral converges. Next we need to determine if the integral test applies to the given series. To apply, the integrand must be positive, continuous, and decreasing. It is positive since xe^{-x^2} is positive for all $x > 0$. It is a rational function of continuous functions, so continuous. We just need to show that it is decreasing, i.e., that $f'(x) < 0$ if $f(x) = xe^{-x^2}$.

$$
\begin{aligned}
f(x) &= xe^{-x^2} \\
f'(x) &= e^{-x^2} + x\left(e^{-x^2}\right)(-2x) \\
&= \frac{1}{e^{x^2}} - \frac{2x^2}{e^{x^2}} \\
&= \frac{1-2x^2}{e^{x^2}}
\end{aligned}
$$

which is negative for all x greater than or equal to 1. Therefore, the function is decreasing for $x \geq 1$. The function is positive, continuous, and decreasing, so the integral test applies and the improper integral and the series converge or diverge together. The integral converges, therefore the $\boxed{\textbf{series converges}}$.

11. (8.3 #20) $A_k = e^{-k}$.

$$\sum_{k=1}^{\infty} A_k = \sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$$

Geometric series with $r = \frac{1}{e} < 1$, so the **series converges**. Converges to:

$$\begin{aligned} S &= \frac{1/e}{1 - 1/e} \\ &= \boxed{\frac{1}{e-1}} \end{aligned}$$

$$\lim_{k \to \infty} \frac{1}{e^k} = \boxed{0}$$

12. (8.3 #24) $A_k = 5/\sqrt{k}$

$$\sum_{k=1}^{\infty} A_k = \sum_{k=1}^{\infty} 5 \cdot \frac{1}{k^{1/2}}$$

This is a p-series with $p = \frac{1}{2} \leq 1$. Therefore the $\boxed{\text{series diverges}}$.

$$\lim_{k \to \infty} 5 \cdot \frac{1}{k^{1/2}} = \boxed{0}$$