Math 1352-11 — WW06 Solutions

October 25, 2008

Assigned problems: 7.5 – 2, 12, 14, 20, 66; 7.6 – 4, 6, 8; 7.7 – wwint3, 4, 12, 20, 36

Always read through the solution sets even if your answer was correct.

Note that with these and many of the integrals in this course, there is more than one way to do the problems. Your solution may be correct even if you used a different method than what I chose. If so, your answer should be mathematically the same as mine, although it can take some algebra and trig identities to see it. If you ever have questions about the method you tried and whether it is correct, please ask.

1. (7.5 #2)

$$\begin{aligned}
\int \frac{2x+3}{\sqrt{x^2+3x}}\,dx &= \int \frac{du}{\sqrt{u}} \\
&\quad \textit{(Where we let } u = x^2+3x \textit{ and } du = (2x+3)dx\textit{)} \\
&= 2u^{1/2} + C \\
&= \boxed{2\sqrt{x^2+3x} + C}
\end{aligned}$$

2. (7.5 #12) First note that $\sin 2x = 2\sin x \cos x$, so we have

$$\begin{aligned}
\int \frac{\sin 2x}{1+\sin^4 x}\,dx &= \int \frac{2\sin x\cos x}{1+\sin^4 x}\,dx \\
&\quad \textit{(Now let } u = \sin^2 x \textit{ and } du = 2\sin x\cos x\,dx\textit{)} \\
&= \int \frac{du}{1+u^2} \\
&= \tan^{-1} u + C \\
&= \boxed{\tan^{-1}(\sin^2 x) + C}
\end{aligned}$$

3. (7.5 #14) We'll use a trig substitution letting $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$:

$$\begin{aligned}
\int \frac{3x+2}{\sqrt{4-x^2}}\,dx &= \int \frac{(3\cdot 2\sin\theta + 2)}{\sqrt{4-4\sin^2\theta}}(2\cos\theta)\,d\theta \\
&= \int \frac{12\sin\theta\cos\theta}{\sqrt{4\cos^2\theta}}\,d\theta + \int \frac{4\cos\theta}{\sqrt{4\cos^2\theta}}\,d\theta \\
&= 6\int \frac{\sin\theta\cos\theta}{\cos\theta}\,d\theta + 2\int \frac{\cos\theta}{\cos\theta}\,d\theta \\
&= 6\int \sin\theta\,d\theta + 2\int d\theta \\
&= -6\cos\theta + 2\theta + C \\
&\quad \textit{(From the substitution, } \sin\theta = x/2\textit{, therefore } \cos\theta = \frac{1}{2}\sqrt{4-x^2} \textit{ and } \theta = \sin^{-1}(x/2).\textit{)} \\
&= \boxed{-3\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + C}
\end{aligned}$$

4. (7.5 #20)

$$\begin{aligned}
\int \frac{dx}{4-e^{-x}} &= \int \frac{dx}{4-e^{-x}}\left(\frac{e^x}{e^x}\right) \\
&= \int \frac{e^x}{4e^x-1}\,dx \\
&\quad \textit{(Let } u = 4e^x - 1 \textit{ and } du = 4e^x\,dx\textit{)} \\
&= \frac{1}{4}\int \frac{du}{u} \\
&= \frac{1}{4}\ln|u| + C \\
&= \boxed{\tfrac{1}{4}\ln|4e^x-1| + C}
\end{aligned}$$

5. (7.5 #66)

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$$

Nothing cancels and we don't have a simple substitution, so try partial fractions.

$$\begin{aligned}
\frac{5x^2+3x-2}{x^3+2x^2} &= \frac{A_1x+B_1}{x^2} + \frac{A_2}{x+2} \\
5x^2+3x-2 &= (A_1x+B_1)(x+2) + A_2x^2 \\
&= (A_1+A_2)x^2 + (2A_1+B_1)x + 2B_1
\end{aligned}$$

Now we equate coefficients of each degree of x giving us equations for A_1, B_1, A_2:

$$\begin{aligned}
A_1 + A_2 &= 5 \\
2A_1 + B_1 &= 3 \\
B_1 &= -1
\end{aligned}$$

Solving these equations gives $A_1 = 2$, $B_1 = -1$, and $A_2 = 3$. So we have:

$$\begin{aligned}
\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx &= \int \frac{2x-1}{x^2} + \frac{3}{x+2}\,dx \\
&= \int \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\,dx \\
&= \boxed{2\ln|x| + \tfrac{1}{x} + 3\ln|x+2| + C}
\end{aligned}$$

6. (7.6 #4)

$$\begin{aligned}
x^2\frac{dy}{dx} + xy &= 2 \\
\frac{dy}{dx} + \frac{y}{x} &= \frac{2}{x^2} \\
\frac{dy}{dx} + P(x)y &= Q(x)
\end{aligned}$$

where $P(x) = 1/x$ and $Q(x) = 2x^{-2}$.

$$\begin{aligned}
I(x) &= e^{\int P(x)\,dx} \\
&= e^{\int \frac{1}{x}\,dx} \\
&= e^{\ln x} \\
&= x
\end{aligned}$$

$$\begin{aligned}
y(x) &= \frac{1}{I(x)}\left[\int Q(x)I(x)\,dx + C\right] \\
&= \frac{1}{x}\left[\int 2x^{-2}\cdot x\,dx + C\right] \\
&= \frac{1}{x}\left[\int 2x^{-1}\,dx + C\right] \\
&= \frac{1}{x}\left[2\ln|x| + C\right] \\
&= \boxed{\frac{2\ln|x|}{x} + \frac{C}{x}}
\end{aligned}$$

7. (7.6 #6)

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$$

$P(x) = \frac{2x+1}{x}$ and $Q(x) = e^{-2x}$

$$\begin{aligned}
I(x) &= e^{\int P(x)\,dx} \\
&= e^{\int \frac{2x+1}{x}\,dx} \\
&= e^{\int 2+\frac{1}{x}\,dx} \\
&= e^{2x+\ln x} \\
&= e^{2x}e^{\ln x} \\
&= xe^{2x}
\end{aligned}$$

$$\begin{aligned}
y(x) &= \frac{1}{I(x)}\left[\int Q(x)I(x)\,dx + C\right] \\
&= x^{-1}e^{-2x}\left[\int e^{-2x}xe^{2x}\,dx + C\right] \\
&= x^{-1}e^{-2x}\left[\int x\,dx + C\right] \\
&= x^{-1}e^{-2x}\left[\frac{1}{2}x^2 + C\right] \\
&= \frac{1}{2}xe^{-2x} + Cx^{-1}e^{-2x} \\
&= \boxed{\frac{x}{2e^{2x}} + \frac{C}{xe^{2x}}}
\end{aligned}$$

8. (7.6 #8)

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\ln x}{x}$$

$P(x) = \frac{2}{x}$ and $Q(x) = \frac{\ln x}{x}$

$$\begin{aligned}
I(x) &= e^{\int P(x)\,dx} \\
&= e^{\int \frac{2}{x}\,dx} \\
&= e^{2\ln x} \\
&= \left(e^{\ln x}\right)^2 \\
&= x^2
\end{aligned}$$

$$\begin{aligned}
y(x) &= \frac{1}{I(x)}\left[\int Q(x)I(x)\,dx + C\right] \\
&= x^{-2}\left[\int x^2\frac{\ln x}{x}\,dx + C\right] \\
&= x^{-2}\left[\int x\ln x\,dx + C\right]
\end{aligned}$$

(We'll use integration by parts on the integral. Let $u = \ln x$ and $dv = x\,dx$, giving $du = \frac{1}{x}\,dx$ and $v = \frac{1}{2}x^2$)

$$\begin{aligned}
&= x^{-2}\left[\frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx + C\right] \\
&= x^{-2}\left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C\right] \\
&= \boxed{\tfrac{1}{2}\ln x - \tfrac{1}{4} + \tfrac{C}{x^2}}
\end{aligned}$$

9. (7.7 webwork)

9.1. This integral is improper since the upper limit of integration is ∞. So we need to decide if it converges or diverges.

$$\begin{aligned}
\int_1^\infty se^{4s^2}\,ds &= \lim_{N\to\infty}\int_1^N se^{4s^2}\,ds \\
&\quad \textit{(let } u = 4s^2 \textit{ and } du = 8s\,ds\textit{)} \\
&= \lim_{N\to\infty}\int_{s=1}^{s=N}\frac{1}{8}e^u\,du \\
&= \lim_{N\to\infty}\frac{1}{8}e^u\Big|_{s=1}^{s=N} \\
&= \lim_{N\to\infty}\frac{1}{8}e^{4s^2}\Big|_1^N \\
&= \lim_{N\to\infty}\frac{1}{8}e^{4N^2}-\frac{1}{8}e^4\to\infty
\end{aligned}$$

The limit goes to infinity, so the **improper integral diverges**. $\boxed{\text{D}}$

9.2. This integral is improper because the integrand is unbounded at $x = 4$. So we need to split into two improper integrals.

$$\begin{aligned}
\int_0^9\frac{1}{\sqrt[3]{x-4}}\,dx &= \int_0^4\frac{dx}{\sqrt[3]{x-4}}+\int_4^9\frac{dx}{\sqrt[3]{x-4}} \\
&= \lim_{t\to4^-}\int_0^t\frac{dx}{\sqrt[3]{x-4}}+\lim_{t\to4^+}\int_t^9\frac{dx}{\sqrt[3]{x-4}} \\
&= \lim_{t\to4^-}\frac{3}{2}(x-4)^{2/3}\Big|_0^t+\lim_{t\to4^+}\frac{3}{2}(x-4)^{2/3}\Big|_t^9 \\
&= \lim_{t\to4^-}\left(\cancelto{0}{\frac{3}{2}(t-4)^{2/3}}-\frac{3}{2}(0-4)^{2/3}\right)+\lim_{t\to4^+}\left(\frac{3}{2}(9-4)^{2/3}-\cancelto{0}{\frac{3}{2}(t-4)^{2/3}}\right) \\
&= \textit{(The left and right limits both go to zero.)} \\
&= -\frac{3}{2}\sqrt[3]{16}+\frac{3}{2}\sqrt[3]{25}
\end{aligned}$$

Both improper integrals converge, so the original **improper integral converges**. $\boxed{\text{C}}$

9.3. This integral is improper since both limits of integration are unbounded. The integrand is bounded at $x = 0$, so we will split the integration at that point.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x}{x^2+5}\,dx &= \int_{-\infty}^{0} \frac{x}{x^2+5}\,dx + \int_{0}^{+\infty} \frac{x}{x^2+5}\,dx \\
&= \lim_{t\to-\infty} \int_{t}^{0} \frac{x}{x^2+5}\,dx + \lim_{N\to\infty} \int_{0}^{N} \frac{x}{x^2+5}\,dx \\
&= \textit{(Let } u = x^2+5 \textit{ and } du = 2x\,dx.) \\
&= \lim_{t\to-\infty} \frac{1}{2} \int_{x=t}^{x=0} \frac{du}{u} + \lim_{N\to\infty} \frac{1}{2} \int_{t=0}^{t=N} \frac{du}{u} \\
&= \lim_{t\to-\infty} \frac{1}{2} \ln|u| \Big|_{x=t}^{x=0} \lim_{N\to\infty} \frac{1}{2} \ln|u| \Big|_{t=0}^{t=N} \\
&= \lim_{t\to-\infty} \frac{1}{2} \ln|x^2+5| \Big|_{t}^{0} \lim_{N\to\infty} \frac{1}{2} \ln|x^2+5| \Big|_{0}^{N}
\end{aligned}$$

Both limits go to infinity, therefore the original **improper integral diverges**. In order to converge, both limits would have to converge. $\boxed{\text{D}}$

9.4.

$$\int_{-\pi/4}^{9\pi/2} \tan^2(4x)\,dx$$

This integral is improper because $\tan\theta$ is unbounded whenever $\cos\theta = 0$, and the limits of integration include such values of θ. To do this integral, we would need to split the integral at every x value where $\tan(4x)$ is unbounded. All of these integrals would need to converge for the original improper integral to converge. So we will look at the first interval. Starting at the lower limit $x = -\frac{\pi}{4}$, the first value of x for which $\tan(4x)$ is unbounded is at $x = \frac{\pi}{8}$ which gives $\tan(\pi/2)$ which is unbounded.

$$\begin{aligned}
\int_{-\pi/4}^{\pi/8} \tan^2(4x)\,dx &= \lim_{N\to\frac{\pi}{8}^-} \int_{-\pi/4}^{N} \tan^2(4x)\,dx \quad \textit{(We'll used Formula 404 with } a = 4 \textit{ and } u = x.) \\
&= \lim_{N\to\frac{\pi}{8}^-} \left[\frac{\tan(4x)}{4} - x\right]\Big|_{-\pi/4}^{N} \\
&= \lim_{N\to\frac{\pi}{8}^-} \left[\frac{\tan(4N)}{4} - N\right] - \left[\frac{\tan(-\pi)}{4} + \frac{\pi}{4}\right] \\
&= \left[\infty - \frac{\pi}{8}\right] - \left[0 + \frac{\pi}{4}\right] \\
&= \infty
\end{aligned}$$

This limit diverges. Therefore, this one improper integral on $[-\pi/4, \pi/8]$ diverges and therefore the entire original **improper integral diverges**. $\boxed{\text{D}}$

9.5.

$$\int_4^9 \ln(x-4)\,dx$$

This integral is improper because the integrand $\ln(x-4)$ is unbounded at one of the limits of integration, $x = 4$.

$$\begin{aligned}
\int_4^9 \ln(x-4)\,dx &= \lim_{t\to 4^+} \int_t^9 \ln(x-4)\,dx \\
&= \lim_{t\to 4^+} \left[(x-4)\ln(x-4) - (x-4)\right]\Big|_t^9 \\
&= 5\ln 5 - 5 - \lim_{t\to 4^+} \left[(t-4)\ln(t-4) - \cancelto{0}{(t-4)}\right] \\
&= 5\ln 5 - 5 + 0 - \lim_{t\to 4^+} \left[(t-4)\ln(t-4)\right]
\end{aligned}$$

(This limit has indeterminate form $0\cdot(-\infty)$. We'll rewrite it to have form $\frac{-\infty}{\infty}$ so we can apply L'hopital's rule.)

$$\begin{aligned}
&= 5\ln 5 - 5 - \lim_{t\to 4^+} \left[\frac{\ln(t-4)}{(t-4)^{-1}}\right] \quad \textit{(This limit now has form } \frac{-\infty}{\infty}\textit{; use L'hopital)} \\
&= 5\ln 5 - 5 - \lim_{t\to 4^+} \cancelto{0}{-(t-4)} \\
&= 5\ln 5 - 5
\end{aligned}$$

Therefore, this **improper integral converges** to $5\ln 5 - 5$. $\boxed{\text{C}}$

9.6.

$$\int_4^\infty \frac{1}{\sqrt{t^2-16}}\,dt$$

This integral is improper since the upper limit of integration is unbounded and the integrand is unbounded at the lower limit of integration. We'll try splitting it up at an unspecified value of $M > 4$ and see if the split integral converges.

$$\begin{aligned}
\int_M^\infty \frac{1}{\sqrt{t^2-16}}\,dt &= \lim_{N\to\infty}\int_M^N \frac{1}{\sqrt{t^2-16}}\,dt \\
&\quad \textit{(Let } u = 4\sec\theta \textit{ and } du = 4\sec\theta\tan\theta\textit{)} \\
&= \lim_{N\to\infty}\int_M^N \frac{4\sec\theta\tan\theta\,d\theta}{\sqrt{16\sec^2\theta-16}} \\
&= \lim_{N\to\infty}\int_M^N \frac{\cancel{4}\sec\theta\cancel{\tan\theta}\,d\theta}{\cancel{4}\cancel{\tan\theta}} \\
&= \lim_{N\to\infty}\int_M^N \sec\theta\,d\theta \quad \textit{(Use Formula 425.)} \\
&= \lim_{N\to\infty} \ln|\sec\theta+\tan\theta|\Big|_{t=M}^{t=N} \\
&\quad \textit{(Note that since } \sec\theta = \frac{t}{4}\textit{, we know that } \tan\theta = \frac{1}{4}\sqrt{t^2-16}\textit{)} \\
&= \lim_{N\to\infty} \ln\left|\frac{t}{4}+\frac{1}{4}\sqrt{t^2-16}\right|\Bigg|_{t=M}^{t=N} \\
&= \lim_{N\to\infty}\left[\ln\left|\frac{N}{4}+\frac{1}{4}\sqrt{N^2-16}\right|\right] - \ln\left|\frac{M}{4}+\frac{1}{4}\sqrt{M^2-16}\right|
\end{aligned}$$

The limit goes to ∞, so the integral diverges. This integral diverges regardless of the value of M. Therefore the original improper integral diverges. $\boxed{\text{D}}$

9.7.

$$\int_{-\infty}^\infty \sin 4x\,dx$$

This integral is improper because the limits of integration are unbounded. We'll try splitting at $x = 0$.

$$\begin{aligned}
\int_{-\infty}^\infty \sin 4x\,dx &= \int_{-\infty}^0 \sin 4x\,dx + \int_0^\infty \sin 4x\,dx \\
&= \lim_{t\to-\infty}\int_t^0 \sin 4x\,dx + \lim_{N\to\infty}\int_0^N \sin 4x\,dx \\
&= \lim_{t\to-\infty} -\frac{1}{4}\cos 4x\,\Big|_t^0 + \lim_{N\to\infty} -\frac{1}{4}\cos 4x\,\Big|_0^N
\end{aligned}$$

The limits diverge. In order for the original integral to converge, both of these integrals would have to converge. Therefore, the original **improper integral diverges**. $\boxed{\text{D}}$

9.8.

$$\int_{-5\pi}^{45\pi} \sin x \tan^{-1} x \, dx$$

The limits of integration are both bounded and the integrand is bounded for all values of x, so the integral is proper. $\boxed{\text{P}}$

10. (7.7 #4)

$$\begin{aligned}
\int_1^\infty \frac{dx}{\sqrt[3]{x}} &= \lim_{N\to\infty} \int_1^N x^{-1/3} \, dx \\
&= \lim_{N\to\infty} \frac{3}{2} x^{2/3} \Big|_1^N \\
&= \lim_{N\to\infty} \frac{3}{2} N^{2/3} - \frac{3}{2}
\end{aligned}$$

The limit goes to ∞, therefore the improper integral **diverges**.

11. (7.7 #12)

$$\begin{aligned}
\int_0^\infty e^{-x} \, dx &= \lim_{N\to\infty} \int_0^N e^{-x} \, dx \\
&= \lim_{N\to\infty} -e^{-x} \Big|_0^N \\
&= \lim_{N\to\infty} -\cancelto{0}{\frac{1}{e^N}} + \frac{1}{e^0} \\
&= 1
\end{aligned}$$

$\boxed{\text{Converges to 1.}}$

12. (7.7 #20)

$$\begin{aligned}
\int_0^\infty xe^{-x}\,dx &= \lim_{N\to\infty}\int_0^N xe^{-x}\,dx \\
&\quad \textit{(Integrate by parts. Let } u=x \textit{ and } dv=e^{-x}\,dx\textit{, which gives } du=dx \textit{ and } v=-e^{-x}\textit{.)} \\
&= \lim_{N\to\infty}\left[-xe^{-x}\Big|_0^N+\int_0^N e^{-x}\,dx\right] \\
&= \lim_{N\to\infty}\left[-Ne^{-N}-0-e^{-x}\Big|_0^N\right] \\
&= \lim_{N\to\infty}\left[-\frac{N}{e^N}-0-\frac{1}{e^N}+\frac{1}{e^0}\right] \\
&= \lim_{N\to\infty}-\frac{N}{e^N}-\lim_{N\to\infty}\frac{1}{e^N}+1 \quad \textit{(Use L'hopital on first limit } \left(\frac{\infty}{\infty}\right)\textit{.)} \\
&= \lim_{N\to\infty}-\cancelto{0}{\frac{1}{e^N}}-\lim_{N\to\infty}\cancelto{0}{\frac{1}{e^N}}+1 \\
&= 1
\end{aligned}$$

$\boxed{\text{Converges to 1.}}$

13. (7.7 #36)

$$\int_0^2 \frac{dx}{(1-x)^2}$$

The integrand is unbounded at $x=1$, so we need to split up the integral.

$$\begin{aligned}
\int_0^2 \frac{dx}{(1-x)^2} &= \lim_{N\to1^-}\int_0^N\frac{dx}{(1-x)^2}+\lim_{t\to1^+}\int_t^2\frac{dx}{(1-x)^2} \\
&\quad \textit{(Let } u=1-x \textit{ and } du=-dx\textit{)} \\
&= \lim_{N\to1^-}\int_0^N\frac{-du}{u^2}+\lim_{t\to1^+}\int_t^2\frac{-du}{u^2} \\
&= \lim_{N\to1^-}u^{-1}\Big|_0^N+\lim_{t\to1^+}u^{-1}\Big|_t^2 \\
&= \lim_{N\to1^-}\frac{1}{(1-x)}\Big|_0^N+\lim_{t\to1^+}\frac{1}{(1-x)}\Big|_t^2 \\
&= \lim_{N\to1^-}\frac{1}{(1-N)}-1+\lim_{t\to1^+}-1-\frac{1}{(1-t)} \\
&= \lim_{N\to1^-}\cancelto{\infty}{\frac{1}{(1-N)}}+\lim_{t\to1^+}-\cancelto{\infty}{\frac{1}{(1-t)}}-2
\end{aligned}$$

$\boxed{\text{The integral diverges.}}$

