Math 1352-11 — WW05 Solutions
October 22, 2008

Assigned problems: 7.3 – 8, 14, 18, 38, 48; 7.4 – 8, 16, 22

Always read through the solution sets even if your answer was correct.

Note that with these and many of the integrals in this course, there is more than one way to do the problems. Your solution may be correct even if you used a different method than what I chose. If so, your answer should be mathematically the same as mine, although it can take some algebra and trig identities to see it. If you ever have questions about the method you tried and whether it is correct, please ask.

1. (7.3 #3)

$$\int \sin^3 x \ \cos^3 x \, dx$$

Since the powers of $\sin x$ and $\cos x$ are both odd, we can peel off either a $\sin x$ or a $\cos x$. I'll peel off the $\cos x$ term.

$$\begin{aligned}
& \int \sin^3 x \ \cos^3 x \, dx \\
= & \int \sin^3 x \ \cos^2 x \, (\cos x \, dx) \\
= & \int \sin^3 x \, (1 - \sin^2 x)(\cos x \, dx) \quad \textit{(since } 1 - \sin^2 x = \cos^2 x\textit{)} \\
= & \int (\sin^3 x - \sin^5 x)(\cos x \, dx) \\
& \quad \textit{(Let } u = \sin x \textit{ and } du = \cos x \, dx\textit{)} \\
= & \int u^3 - u^5 \, du \\
= & \frac{1}{4} u^4 - \frac{1}{6} u^6 + C \\
= & \boxed{\tfrac{1}{4} \sin^4 x - \tfrac{1}{6} \sin^6 x + C}
\end{aligned}$$

2. (7.3 #14)

$$\int \frac{\sin x}{\cos^5 x}\,dx$$

A simple u substitution works fine for this integral.

Let $u = \cos x$ and $du = -\sin x\,dx$.

$$\begin{aligned}
\int \frac{\sin x}{\cos^5 x}\,dx &= -\int \frac{du}{u^5} \\
&= \frac{1}{4}u^{-4} + C \\
&= \frac{1}{4}\cos^{-4} x + C \\
&= \boxed{\tfrac{1}{4\cos^4 x} + C}
\end{aligned}$$

3. (7.3 #18)

$$\int \sec^5 x \tan x\,dx$$

Since the $\tan x$ term is raised to an odd power (1), we peel off $\sec x \tan x$ and use the substitution $u = \sec x$.

$$\begin{aligned}
&\int \sec^5 x \tan x\,dx \\
=\ &\int \sec^4 x(\sec x \tan x\,dx) \qquad \textit{(Let } u = \sec x \textit{ and } du = \sec x \tan x\,dx\textit{)} \\
=\ &\int u^4\,du \\
=\ &\frac{1}{5}u^5 + C \\
=\ &\boxed{\tfrac{1}{5}\sec^5 x + C}
\end{aligned}$$

4. (7.3 #36)

$$\int \sqrt{9+x^2}\,dx$$

Let $x = 3\tan\theta$; then $dx = 3\sec^2\theta\,d\theta$.

$$\begin{aligned}
&\int \sqrt{9+x^2}\,dx \\
=& \int 3\sqrt{9+9\tan^2\theta}\ \sec^2\theta\,d\theta \\
=& \int 3\sqrt{9(1+\tan^2\theta)}\ \sec^2\theta\,d\theta \\
=& 3\int \sqrt{9\sec^2\theta}\ \sec^2\theta\,d\theta \quad \textit{(Since } 1+\tan^2\theta=\sec^2\theta\textit{)} \\
=& 9\int \sec^3\theta\,d\theta \quad \textit{(Use Formula 428 for this integral)} \\
=& 9\left[\frac{\sec\theta\tan\theta}{2}\right] + 9\left(\frac{1}{2}\right)\int \sec\theta\,d\theta \quad \textit{(Now use Formula 425 on the remaining integral)} \\
=& 9\left[\frac{\sec\theta\tan\theta}{2}\right] + \frac{9}{2}\ln|\sec\theta+\tan\theta| + C
\end{aligned}$$

We're almost done now, but we have to convert back to $x's$ from $\theta's$. We know from our substitution that $x = 3\tan\theta$. Therefore, $tan\theta = \frac{x}{3}$. Recall that tangent is opposite over adjacent, so a representative triangle with angle θ has legs of length x and 3. Using the Pythagorean theorem, we see that the hypotenuse of the triangle is then $\sqrt{x^2+9}$. Since $\sec\theta$ is hypotenuse over adjacent (i.e., 1 $\cos\theta$), we have $\sec\theta = \frac{1}{3}\sqrt{x^2+9}$. Therefore, the integral is:

$$\begin{aligned}
\int \sqrt{9+x^2}\,dx &= \frac{9}{2}(\sec\theta\tan\theta) + \frac{9}{2}\ln|\sec\theta+\tan\theta| + C \\
&= \frac{9}{2}\left[\left(\frac{1}{3}\sqrt{x^2+9}\right)\left(\frac{x}{3}\right)\right] + \frac{9}{2}\ln\left|\left(\frac{1}{3}\sqrt{x^2+9}\right)+\left(\frac{x}{3}\right)\right| + C \\
&= \boxed{\tfrac{1}{2}x\sqrt{x^2+9} + \tfrac{9}{2}\ln\left|\left(\tfrac{1}{3}\sqrt{x^2+9}\right)+\left(\tfrac{x}{3}\right)\right| + C}
\end{aligned}$$

5. (7.3 #48)

$$\int \frac{dx}{\sqrt{x^2+8x+3}}$$

The denominator looks almost like the form $\sqrt{u^2 \pm a^2}$ if we could factor it. It doesn't factor simply, so we need to complete the square:

$$\begin{aligned}
x^2+8x+3 &= (x^2+8x\quad)+3 \\
&= (x^2+8x+16)+3-16 \quad \textit{(We have added zero, } 16-16\textit{, so we still have equality)} \\
&= (x+4)^2-13
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+8x+3}} &= \int \frac{dx}{\sqrt{(x+4)^2-13}} \\
&= \int \frac{dx}{\sqrt{(x+4)^2-13}} \quad \textit{(Let } u = x+4 \textit{ and } du = dx\textit{)} \\
&= \int \frac{du}{\sqrt{u^2-13}} \quad \textit{(Now let } u = \sqrt{13}\sec\theta \textit{ and } du = \sqrt{13}\sec\theta\tan\theta\, d\theta\textit{)} \\
&= \int \frac{1}{\sqrt{13\sec^2\theta - 13}} \sqrt{13}\sec\theta\tan\theta\, d\theta \\
&= \int \frac{1}{\sqrt{13(\sec^2\theta - 1)}} \sqrt{13}\sec\theta\tan\theta\, d\theta \\
&= \int \frac{1}{\sqrt{13(\tan^2\theta)}} \sqrt{13}\sec\theta\tan\theta\, d\theta \quad \textit{(since } \tan^2\theta = \sec^2\theta - 1\textit{)} \\
&= \int \frac{1}{\cancel{\sqrt{13}\tan\theta}} \cancel{\sqrt{13}}\sec\theta\cancel{\tan\theta}\, d\theta \\
&= \int \sec\theta\, d\theta \quad \textit{(Use Formula 425)} \\
&= \ln|\sec\theta + \tan\theta| + C \\
&\qquad \textit{(}u = \sqrt{13}\sec\theta\textit{, so } \sec\theta = \frac{u}{\sqrt{13}} \textit{ and } \tan\theta = \frac{\sqrt{u^2-13}}{\sqrt{13}}\textit{)} \\
&= \ln\left|\frac{u}{\sqrt{13}} + \frac{\sqrt{u^2-13}}{\sqrt{13}}\right| + C \\
&\quad \textit{(Now recall that } u = x+4.\textit{)} \\
&= \ln\left|\frac{x+4}{\sqrt{13}} + \frac{\sqrt{(x+4)^2-13}}{\sqrt{13}}\right| + C \\
&= \boxed{\ln\left|\tfrac{1}{\sqrt{13}}\left(x+4+\sqrt{x^2+8x+3}\right)\right| + C}
\end{aligned}$$

6. (7.4 #8) The degree of the denominator (3) is greater than the degree of the numerator (2). In the denominator, we have one linear term and one irreducible quadratic term.

$$\begin{aligned}
\frac{x^2-5x-4}{(x-3)(x^2+1)} &= \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-3)} \\
(x^2-5x-4) &= (Ax+B)(x-3) + C(x^2+1) \quad \textit{(multiplied through by common demoninator)} \\
&\quad \textit{(This eqn. is true for any } x\textit{, therefore true for } x=3 \textit{ giving us } C=-1.\textit{)} \\
(x^2-5x-4) &= Ax^2 + Bx - 3Ax - 3B - x^2 - 1 \\
(x^2-5x-4) &= (A-1)x^2 + (B-3A)x - 3B - 1
\end{aligned}$$

Now we equate coefficients of each degree of x giving us equations for A, B, C.

$$\begin{aligned} A - 1 &= 1 \\ B - 3A &= -5 \\ -3B - 1 &= -4 \end{aligned}$$

Solving these equations gives us $A = 2$ and $B = 1$. Therefore,

$$\boxed{\frac{x^2-5x-4}{(x-3)(x^2+1)} = \frac{2x+1}{(x^2+1)} - \frac{1}{(x-3)}}$$

7. (7.4 #16)

$$\int \frac{(3x-1)}{(x^2-1)}\,dx$$

We don't have a simple u substitution here, so we'll try partial fractions to get two fractions that we can integrate. So first we rewrite the fraction using partial fractions:

$$\begin{aligned} \frac{(3x-1)}{(x^2-1)} &= \frac{(3x-1)}{(x+1)(x-1)} \quad \textit{(Factor the denominator.)} \\ \frac{(3x-1)}{(x^2-1)} &= \frac{A}{(x+1)} + \frac{B}{(x-1)} \\ 3x - 1 &= A(x-1) + B(x+1) \quad \textit{(multiply through by common denominator)} \\ 3x - 1 &= (A+B)x + (B-A) \end{aligned}$$

Now we equate coefficients of each degree of x giving us equations for A and B.

$$\begin{aligned} A + B &= 3 \\ B - A &= -1 \end{aligned}$$

Solving these two equations gives us $\boxed{A = 2 \text{ and } B = 1}$.

Now we can do the integral:

$$\begin{aligned} \int \frac{(3x-1)}{(x^2-1)}\,dx &= \int \frac{2}{x+1} + \frac{1}{x-1}\,dx \\ &= \boxed{2\ln|x+1| + \ln|x-1| + C} \end{aligned}$$

8. (7.4 #22)

$$\int \frac{x^4 - x^2 + 2}{x^2(x-1)}\,dx$$

The degree of the numerator (4) is larger than the degree of the denominator (3), so our first step is to use synthetic division to get

$$\frac{x^4 - x^2 + 2}{x^2(x-1)} = (x+1) + \frac{2}{x^3 - x^2}$$

We can integrate the $x+1$ part easily. We'll need to do partial fractions on the remainder term:

$$\begin{aligned}
\frac{2}{x^3 - x^2} &= \frac{A_1 x + B_1}{x^2} + \frac{A_2}{(x-1)} \\
2 &= (A_1 x + B_1)(x-1) + A_2 x^2 \\
2 &= (A_1 + A_2)x^2 + (B_1 - A_1)x - B_1
\end{aligned}$$

Now we equate coefficients of each degree of x giving us equations for A_1, B_1, A_2.

$$\begin{aligned}
A_1 + A_2 &= 0 \\
B_1 - A_1 &= 0 \\
-B_1 &= 2
\end{aligned}$$

Solving these two equations gives us $A_1 = -2$, $B_1 = -2$, and $A_2 = 2$.

Now we can put all of this together and actually solve the integral:

$$\begin{aligned}
\int \frac{x^4 - x^2 + 2}{x^2(x-1)}\,dx &= \int (x+1) + \frac{-2x-2}{x^2} + \frac{2}{x-1}\,dx \\
&= \int x + 1 - \frac{2}{x} - \frac{2}{x^2} + \frac{2}{x-1}\,dx \\
&= \boxed{\tfrac{1}{2}x^2 + x - 2\ln|x| + 2x^{-1} + 2\ln|x-1| + C}
\end{aligned}$$