
Math 1352-11 — WW03 Solutions
September 21, 2008

Assigned problems: 6.4 – 2, 6, 10, 14, 18, 20, 24; 6.5 – 4, 6, 12

Always read through the solution sets even if your answer was correct.

- (6.4 #2) Arc length of curve $y = 5 - 4x$ on $[-2, 0]$.
 $f'(x) = -4$.

$$\begin{aligned} s &= \int_{-2}^0 \sqrt{1 + (f'(x))^2} \, dx \\ &= \int_{-2}^0 \sqrt{1 + (-4)^2} \, dx \\ &= \int_{-2}^0 \sqrt{17} \, dx \\ &= \sqrt{17}x \Big|_{-2}^0 \\ &= 0 - (-2)\sqrt{17} \\ &= \boxed{2\sqrt{17}} \end{aligned}$$

- (6.4 # 6) Arc length of curve $y = \frac{1}{2}(2 + x^2)^{3/2}$ on $[0, 3]$.
Using the chain rule, we have $f'(x) = \frac{1}{2}(2 + x^2)^{1/2}(2x) = x(2 + x^2)^{1/2}$.

$$\begin{aligned} s &= \int_0^3 \sqrt{1 + (f'(x))^2} \, dx \\ &= \int_0^3 \sqrt{1 + x^2(2 + x^2)} \, dx \\ &= \int_0^3 \sqrt{1 + 2x^2 + x^4} \, dx \\ &= \int_0^3 \sqrt{(1 + x^2)^2} \, dx \\ &= \int_0^3 1 + x^2 \, dx \\ &= x + \frac{1}{3}x^3 \Big|_0^3 \\ &= (3 + 9) - 0 \\ &= \boxed{12} \end{aligned}$$

- (6.4 # 10) Arc length of curve $y = \sqrt{e^{2x} - 1} - \sec^{-1}(e^x)$ on $[0, \ln 2]$.
Using the chain rule and some simplifying, we get:

$$\begin{aligned} f'(x) &= \frac{1}{2}(e^{2x} - 1)^{-1/2}(2e^{2x}) - \frac{1}{e^x \sqrt{e^{2x} - 1}} e^x \\ &= \frac{e^{2x}}{\sqrt{e^{2x} - 1}} - \frac{1}{\sqrt{e^{2x} - 1}} \\ &= \frac{e^{2x} - 1}{\sqrt{e^{2x} - 1}} \\ &= \sqrt{e^{2x} - 1} \end{aligned}$$

So the arc length is:

$$\begin{aligned} s &= \int_0^{\ln 2} \sqrt{1 + (f'(x))^2} dx \\ &= \int_0^{\ln 2} \sqrt{1 + (\sqrt{e^{2x} - 1})^2} dx \\ &= \int_0^{\ln 2} \sqrt{1 + e^{2x} - 1} dx \\ &= \int_0^{\ln 2} \sqrt{e^{2x}} dx \\ &= \int_0^{\ln 2} e^x dx \\ &= e^x \Big|_0^{\ln 2} \\ &= e^{\ln 2} - e^0 = 2 - 1 = \boxed{1} \end{aligned}$$

- (6.4 # 14) Surface area generated when function $f(x) = \sqrt{x}$ on the interval $[2, 6]$ is revolved about the x-axis. $f'(x) = \frac{1}{2}x^{-1/2}$. Surface area, SA, is given by:

$$\begin{aligned}
 SA &= 2\pi \int_2^6 f(x) \sqrt{1 + (f'(x))^2} dx \\
 &= 2\pi \int_2^6 x^{1/2} \sqrt{1 + \left(\frac{1}{2}x^{-1/2}\right)^2} dx \\
 &= 2\pi \int_2^6 x^{1/2} \sqrt{1 + \frac{1}{4}x^{-1}} dx \\
 &= 2\pi \int_2^6 \sqrt{x + \frac{1}{4}} dx \quad (\text{bringing } x^{1/2} \text{ inside the square root}) \\
 &= 2\pi \int_{x=2}^{x=6} u^{1/2} du \quad (\text{letting } u = x + \frac{1}{4} \text{ and } du = dx) \\
 &= 2\pi \frac{2}{3} u^{3/2} \Big|_{x=2}^{x=6} \\
 &= \frac{4\pi}{3} \left(x + \frac{1}{4}\right)^{3/2} \Big|_2^6 \\
 &= \frac{4\pi}{3} \left(\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2} \right) \\
 &= \frac{4\pi}{3} \left(\frac{125}{8} - \frac{27}{8} \right) \\
 &= \boxed{\frac{49\pi}{3}}
 \end{aligned}$$

- (6.4 # 18) Arc length of polar curve $r = \sin \theta + \cos \theta$.

First we need to determine the angles swept through by θ to form the curve. One way to do this is to graph the curve. I'll convert to Cartesian coordinates to graph:

$$\begin{aligned}
 r &= \sin \theta + \cos \theta \\
 r^2 &= r \sin \theta + r \cos \theta \quad (\text{mult. both sides by } r) \\
 x^2 + y^2 &= x + y \quad (\text{since } x = r \cos \theta, y = r \sin \theta, \text{ and } x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2) \\
 0 &= x^2 - x + y^2 - y \\
 \frac{1}{4} + \frac{1}{4} &= x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} \quad (\text{completing the square}) \\
 \frac{1}{2} &= \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2
 \end{aligned}$$

This is the standard form for a circle centered at $(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$ with radius $\sqrt{\frac{1}{2}}$.

Next we can plot some points in polar coordinates to see how far we need to sweep θ to form the whole circle.

r	θ
0	1
$\pi/4$	$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$
$\pi/2$	$1 + 0 = 1$
$3\pi/4$	$-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$
π	$0 - 1 = -1$

Notice that the final point $(r, \theta) = (-1, \pi) = (1, 0)$, so we have come back around to the beginning of the circle at $\theta = \pi$. Therefore, to find the arc length, we need to integrate from $\theta = 0$ to $\theta = \pi$.

$$\begin{aligned}
 s &= \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
 &= \int_0^\pi \sqrt{(\sin \theta + \cos \theta)^2 + (\cos \theta - \sin \theta)^2} d\theta \\
 &= \int_0^\pi \sqrt{(\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta) + (\cos^2 \theta - 2 \sin \theta \cos \theta + \sin^2 \theta)} d\theta \\
 &= \int_0^\pi \sqrt{2(\sin^2 \theta + \cos^2 \theta)} d\theta \\
 &= \int_0^\pi \sqrt{2} d\theta \\
 &= \sqrt{2}\theta \Big|_0^\pi \\
 &= \boxed{\sqrt{2}\pi}
 \end{aligned}$$

Note that this is the answer we expect since the curve is a circle of radius $\sqrt{2}/2$, meaning that its circumference is $2\pi \text{ rad} = 2\pi\sqrt{2}/2 = \sqrt{2}\pi$.

- (6.4 # 20) Arc length of polar curve $r = e^{1-\theta}$ over $0 \leq \theta \leq 1$.

$$\frac{dr}{d\theta} = -e^{1-\theta}, \left(\frac{dr}{d\theta}\right)^2 = e^{2-2\theta}, \text{ and } r^2 = e^{2-2\theta}.$$

The arc length is then

$$\begin{aligned}
 s &= \int_0^1 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
 &= \int_0^1 \sqrt{e^{2-2\theta} + e^{2-2\theta}} d\theta \\
 &= \int_0^1 \sqrt{2e^{2-2\theta}} d\theta \\
 &= \int_0^1 \sqrt{2} e^{1-\theta} d\theta \\
 &= -\sqrt{2} e^{1-\theta} \Big|_0^1 \\
 &= -\sqrt{2}e^0 - (-\sqrt{2}e^1) = \boxed{\sqrt{2}(e - 1)}
 \end{aligned}$$

- (6.4 # 24) Surface area generated when polar curve $r = 1 - \cos \theta$ on $0 \leq \theta \leq \pi$ is revolved about the x-axis.

$$\begin{aligned}
 SA &= 2\pi \int_0^\pi r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
 &= 2\pi \int_0^\pi (1 - \cos \theta) \sin \theta \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} d\theta \\
 &= 2\pi \int_0^\pi (1 - \cos \theta) \sin \theta \sqrt{1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta \\
 &= 2\pi \int_0^\pi (1 - \cos \theta) \sin \theta \sqrt{2(1 - \cos \theta)} d\theta \\
 &= 2\pi \int_0^\pi (1 - \cos \theta) \sin \theta \sqrt{2} (1 - \cos \theta)^{1/2} d\theta \\
 &= 2\pi \sqrt{2} \int_0^\pi (1 - \cos \theta)^{3/2} \sin \theta d\theta \\
 &\quad (\text{let } u = 1 - \cos \theta \text{ then } du = \sin \theta d\theta) \\
 &= 2\sqrt{2}\pi \int_{\theta=0}^{\theta=\pi} u^{3/2} du \\
 &= 2\sqrt{2}\pi \left. \frac{2}{5} u^{5/2} \right|_{\theta=0}^{\theta=\pi} \\
 &= \frac{4\sqrt{2}\pi}{5} (1 - \cos \theta)^{5/2} \Big|_0^\pi \\
 &= \frac{4\sqrt{2}\pi}{5} \left((2)^{5/2} - (0)^{5/2} \right) = \frac{2^2 2^{1/2} 2^{5/2} \pi}{5} = \frac{2^{10/2} \pi}{5} = \frac{2^5 \pi}{5} = \boxed{\frac{32\pi}{5}}
 \end{aligned}$$

- (6.5 # 4) Work done in lifting a 50 lb bag 5 ft.
This is a constant force, so work is just $W = Fd$, force times distance.

$$\begin{aligned}
 W &= Fd \\
 &= 50 \times 5 \text{ foot-pounds} \\
 &= \boxed{250 \text{ foot-pounds}}
 \end{aligned}$$

- (6.5 # 6) It takes 4 ergs of work to stretch a spring 10 cm past equilibrium. How much work to stretch 4 cm further?
First we need to determine the spring constant k for this spring. We have

$$\begin{aligned}
 W &= \int_0^{10} kx dx \\
 4 &= \int_0^{10} kx dx \\
 4 &= \left. \frac{1}{2} kx^2 \right|_0^{10} = 50k \\
 k &= \frac{2}{25}
 \end{aligned}$$

Now we can determine the amount of work stretching 4 cm further (i.e., from 10 to 14 cm):

$$\begin{aligned} W &= \int_{10}^{14} \frac{2}{25}x \, dx \\ &= \left. \frac{1}{25}x^2 \right|_{10}^{14} \\ &= \frac{1}{25}(14)^2 - \frac{1}{25}(10)^2 \\ &= \frac{196 - 100}{25} \\ &= \boxed{\frac{96}{25} \text{ ergs}} \end{aligned}$$

- (6.5 # 12) Force $F(x) = x^4 + 2x^2$. Work done (in ergs) by force (in dynes) in moving object from $x = 1$ cm to $x = 2$ cm is:

$$\begin{aligned} W &= \int_1^2 x^4 + 2x^2 \, dx \\ &= \left. \frac{1}{5}x^5 + \frac{2}{3}x^3 \right|_1^2 \\ &= \left(\frac{32}{5} + \frac{16}{3} \right) - \left(\frac{1}{5} + \frac{2}{3} \right) \\ &= \left(\frac{96}{15} + \frac{80}{15} - \frac{3}{15} - \frac{10}{15} \right) \\ &= \frac{176 - 13}{15} \\ &= \frac{163}{15} \text{ ergs} \end{aligned}$$