Math 1352-11 — HW05 Solutions

November 20, 2008

Assigned problems: 8.3 – 34, 56; 8.4 – 26, 28; 8.5 – 34, 36; 8.6 – 16, 34

Always read through the solution sets even if your answer was correct.

- (8.3 #34)

$$\sum_{k=1}^{\infty} \frac{k^2}{e^k}$$

Trying the **integral test**:

Let $f(x) = \frac{x^2}{e^x}$. The integral test applies if $f(x)$ is positive, continuous, and decreasing for all $x \geq N$ for some N. The function $f(x)$ is clearly positive and continuous, so we just need to check that it's decreasing. Decreasing if $f'(x) < 0$.

$$\begin{aligned} f(x) &= \frac{x^2}{e^x} = x^2 e^{-x} \\ f'(x) &= 2xe^{-x} + x^2(-1)e^{-x} \\ &= \frac{2x - x^2}{e^x} \end{aligned}$$

So $f'(x) < 0$ if $2x - x^2 < 0$, which is true for all $x > 2$. Therefore $f(x)$ decreasing for all $x > 2$ ("eventually decreasing"). Therefore, the integral test applies. The test then says that if the improper integral $\int_1^\infty f(x)\,dx$ converges.

$$\begin{aligned} \int_1^\infty f(x)\,dx &= \int_1^\infty \frac{x^2}{e^x}\,dx \\ &= \lim_{N\to\infty} \int_1^N \frac{x^2}{e^x}\,dx \\ & \quad \textit{(Int. by parts: let } u = x^2 \textit{ so } du = 2x\,dx, \textit{ and } dv = e^{-x}\,dx, \textit{ so } v = -e^{-x}.\textit{)} \\ &= \lim_{N\to\infty} \left[-x^2 e^{-x}\Big|_1^N + \int_1^N 2xe^{-x}\,dx \right] \\ & \quad \textit{(We'll use int. by parts again on the second integral: Let } u = x \textit{ so } du = dx, \textit{ and} \\ & \quad dv = e^{-x}\,dx \textit{ so } v = -e^{-x}.\textit{)} \\ &= \lim_{N\to\infty} \left[-x^2 e^{-x}\Big|_1^N - 2xe^{-x}\Big|_1^N + \int_1^N 2e^{-x}\,dx \right] \\ &= \lim_{N\to\infty} \left[\left(-x^2 e^{-x} - 2xe^{-x} - 2x^{-x} \right)\Big|_1^N \right] \\ &= \lim_{N\to\infty} \frac{-N^2 - 2N - 2}{e^N} + \frac{1+2+2}{e} = \frac{5}{e} \end{aligned}$$

The integral converges, therefore $\boxed{\textbf{by the integral test, the original series converges}}$.

- (8.3 #56)

$$\sum_{k=2}^{\infty} \frac{k^2}{(k^2+4)^p} \quad = \quad \sum_{k=2}^{\infty} a_k$$

I'll try the **direct comparison test**. Let

$$\sum_{k=2}^{\infty} b_k \quad = \quad \sum_{k=2}^{\infty} \frac{k^2}{k^{2p}} \quad = \quad \sum_{k=2}^{\infty} \frac{1}{k^{2p-2}}$$

Note that $0 < a_k < b_k$, and that $\sum_{k=2}^{\infty} b_k$ is a convergent p-series if $2p-2 > 1$, i.e., $p > 3/2$. So by the direct comparison test, the $\boxed{\textbf{series converges if } p > 3/2}$.

Now, what if $p \le 3/2$? We'll try a limit comparison test with $\sum \frac{1}{k} = \sum c_k$. For the limit comparison test, we examine the limit

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{c_k} &= \lim_{k\to\infty} \frac{k^2/(k^2+4)^{3/2}}{1/k} \\
&= \lim_{k\to\infty} \frac{k^3}{(k^2+4)^{3/2}} \\
&= \lim_{k\to\infty} \frac{3k^2}{\frac{3}{2}(k^2+4)^{1/2}(2k)} \quad \textit{(L'hopital)} \\
&= \lim_{k\to\infty} \frac{k}{(k^2+4)^{1/2}} \\
&= \lim_{k\to\infty} \frac{1}{\frac{1}{2}(k^2+4)^{-1/2}(2k)} \quad \textit{(L'hopital)} \\
&= \lim_{k\to\infty} \frac{\sqrt{k^2+4}}{k} \\
&= \lim_{k\to\infty} \frac{\sqrt{k^2+4}}{k} \cdot \frac{1/k}{1/k} \\
&= \lim_{k\to\infty} \frac{\sqrt{1+4/k^2}}{1} \quad = \quad 1
\end{aligned}$$

The limit is positive and finite, so the limit comparison test says that the two series converge or diverge together. The series $\sum \frac{1}{k}$ is a divergent p-series. Therefore, our $\boxed{\textbf{original series diverges when } p \le 3/2}$.

- (8.4 #26)

$$\sum \frac{3k^2+2}{k^2+3k+2}$$

A series must diverge if it's terms a_k do not go to zero. Since the numerator and denominator have the same degree, I'll try the divergence test first.

$$\begin{aligned} \lim_{k\to\infty} a_k &= \lim_{k\to\infty} \frac{3k^2+2}{k^2+3k+2} \\ &= \lim_{k\to\infty} \frac{6k}{2k+3} \quad \textit{(L'hopital)} \\ &= \lim_{k\to\infty} \frac{6}{2} \quad \textit{(L'hopital)} \\ &= 3 \end{aligned}$$

The limit does not go to zero, therefore the $\boxed{\textbf{series diverges}}$ by the divergence test.

- (8.4 #28)

$$\sum \frac{5}{4^k+3}$$

Trying the **limit comparison test**. We'll compare to $\sum b_k = \sum \frac{5}{4^k}$.

$$\begin{aligned} \lim_{k\to\infty} \frac{5/(4^k+3)}{5/4^k} &= \lim_{k\to\infty} \frac{5\cdot 4^k}{5\cdot(4^k+3)} \\ &= \lim_{k\to\infty} \frac{4^k}{(4^k+3)} \\ &= 1 \end{aligned}$$

The limit is positive and finite, so the two series converge or diverge together. $\sum b_k = \sum \frac{5}{4^k}$ is a convergent geometric series $\sum 5\left(\frac{1}{4}\right)^k$, therefore the $\boxed{\textbf{original series converges}}$.

- (8.5 #34)

$$\sum \frac{2^{2k}k!}{k^k}$$

The presence of k^k and 2^{2k} makes me think of the root test. However, the k^{th} root of $k!$ is too complicated. So the factorial makes me decide to try the **ratio test** instead. For the ratio test, we look at the limit:

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \lim_{k\to\infty} \frac{2^{2k+2}(k+1)!/(k+1)^{k+1}}{2^{2k}k!/k^k} \\
&= \lim_{k\to\infty} \frac{(k^k)(2^{2k+2})(k+1)!}{(k+1)^{k+1}(2^{2k})(k!)} \\
&\quad \textit{(Note that } (k+1)! = (k+1)(k!) \textit{ and that } (k+1)^{k+1} = (k+1)(k+1)^k\textit{.)} \\
&= \lim_{k\to\infty} \frac{2^2 k^k}{(k+1)^k} \\
&= 4 \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k \\
&= 4 \lim_{k\to\infty} \left(\frac{1}{1+1/k}\right)^k \\
&= 4 \lim_{k\to\infty} \frac{1}{(1+1/k)^k} \quad \textit{(Note that the denominator is a limit I will assume you know: } \lim_{k\to\infty}(1+1/k)^k = e\textit{.)} \\
&= \frac{4}{e}
\end{aligned}$$

The value of the limit is greater than 1, therefore according to the ratio test, $\boxed{\textbf{the series diverges}}$.

- (8.5 #36)

$$\sum \frac{1}{k^k}$$

The k^k term should make you think to try the **root test**. So we look at the limit:

$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{1/k^k} &= \lim_{k\to\infty} (1/k^k)^{1/k} \\
&= \lim_{k\to\infty} \frac{1}{k} \\
&= 0
\end{aligned}$$

The limit is $0 < 1$, therefore, by the root test, the $\boxed{\textbf{series converges}}$.

- (8.6 #16)

$$\sum (-1)^{k+1} \frac{2^k}{k!}$$

This is an alternating series. We could try the alternating series test, and I believe that would work. The factorial also makes me think of the **generalized ratio test** ("generalized" because the terms of the series are not all positive), so

I'll try that one:

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{2^{k+1}/(k+1)!}{2^k/k!}\right| \\
&= \lim_{k\to\infty}\frac{2^{k+1}k!}{2^k(k+1)!} \\
&= \lim_{k\to\infty}\frac{2}{k+1} \\
&= 0
\end{aligned}$$

The limit is less than 1, therefore, by the generalized ratio test, $\boxed{\textbf{the series converges}}$.

- (8.6 #34)

$$\sum \frac{(-1)^k}{k^2}$$

(a.) Estimate the sum of this alternating series with the sum of the first 4 terms. How accurate is this estimate? The first 4 terms are the 4th partial sum S_4:

$$\begin{aligned}
S_4 = -\frac{1}{1}+\frac{1}{4}-\frac{1}{9}+\frac{1}{16} &= -\frac{115}{144} \\
&\approx \boxed{-0.7986}
\end{aligned}$$

For accuracy, we know that the error in estimating the value S of an alternating series by its n^{th} partial sum S_n is $\leq$ the absolute value of the $(n+1)^{st}$ term. That is,

$$\begin{aligned}
|S-S_n| &\leq |a_{n+1}| \ \textit{or in this case} \\
|S-S_4| &\leq |a_5| \\
|S-S_4| &\leq \frac{1}{25} = \boxed{0.04}
\end{aligned}$$

(b.) How many terms of the series do we need to take to get at least 3 place accuracy?
In other words, we need to find n so that

$$\begin{aligned}
|S-S_n| &\leq |a_{n+1}| \leq 0.0005 \\
\Rightarrow \frac{1}{(n+1)^2} &\geq \frac{1}{2000} \\
\Rightarrow (n+1)^2 &\leq 2000 \\
n &\leq \sqrt{2000}-1 \\
&\approx 43.72
\end{aligned}$$

Therefore, for the given accuracy requirement, $\boxed{\textbf{we need } n \geq 44}$. So after summing the first 44 terms of the series to get the estimate $S \approx S_{44}$, our error $|S-S_{44}|$ is $\leq |a_{45}| \approx 0.000498$.