Math 1352-11 — HW04 Solutions

October 27, 2008

Assigned problems: 7.4 – 52; 7.5 – 76; 7.6 – 28; 7.7 – 6, 42; 7.8 – 4, 14, 26, 44

Always read through the solution sets even if your answer was correct.

- (7.4 #52)

$$\int \frac{dx}{3\sin x + 4\cos x + 5}$$

This integral is a rational function of sines and cosines, so we'll us a Weierestrass substitution. Let $u = \tan\frac{x}{2}$, then $dx = \frac{2}{1+u^2}\,du$, $\sin x = \frac{2u}{1+u^2}$, and $\cos x \frac{1-u^2}{1+u^2}$.

$$\begin{aligned}
\int \frac{dx}{3\sin x + 4\cos x + 5} &= \int \frac{\frac{2}{1+u^2}\,du}{\frac{6u}{1+u^2} + \frac{4(1-u^2)}{1+u^2} + 5} \\
&= \int \frac{2\,du}{6u + 4 - 4u^2 + 5 + 5u^2} \\
&= \int \frac{2\,du}{u^2 + 6u + 9} \\
&= \int \frac{2\,du}{(u+3)^2} \\
&\quad \textit{(Let } w = u+3,\ dw = du\textit{)} \\
&= \int \frac{2\,dw}{w^2} \\
&= -2w^{-1} + C \\
&= \frac{-2}{u+3} + C \\
&= \boxed{\frac{-2}{\tan\frac{x}{2}+3} + C}
\end{aligned}$$

- (7.5 #76) Arc length of curve $y = \ln x$ from $x = 2$ to $x = 3$.

First recall that arc length of $y = f(x)$ from a to b is given by

$$s = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

Since $y' = 1/x$ we have

$$
\begin{aligned}
s &= \int_a^b \sqrt{1+(f'(x))^2}\,dx \\
&= \int_2^3 \sqrt{1+\frac{1}{x^2}}\,dx \\
&= \int_2^3 \sqrt{\frac{1}{x^2}(x^2+1)}\,dx \\
&= \int_2^3 \frac{\sqrt{x^2+1}}{x}\,dx \\
&\quad \textit{(Let } x = \tan\theta \textit{ and } dx = \sec^2\theta\,d\theta\textit{)} \\
&= \int_2^3 \frac{\sqrt{\tan^2\theta+1}}{\tan\theta}\sec^2\,d\theta \\
&= \int_2^3 \frac{\sec^3\theta}{\tan\theta}\,d\theta \\
&= \int_2^3 \frac{\sec^3\theta\tan\theta}{\tan^2\theta}\,d\theta \\
&= \int_2^3 \frac{\sec^2\theta(\sec\theta\tan\theta\,d\theta)}{\tan^2\theta} \quad \textit{(Now we have an odd power of } \tan x \textit{ so peel off a } \sec\theta\tan\theta\,d\theta\textit{.)} \\
&= \int_2^3 \frac{\sec^2\theta(\sec\theta\tan\theta\,d\theta)}{\sec^2\theta-1} \\
&\quad \textit{(Let } u = \sec\theta,\ du = \sec\theta\tan\theta\,d\theta\textit{.)} \\
&= \int_{x=2}^{x=3} \frac{u^2\,du}{u^2-1} \quad \textit{(Use itegration Formula 76.)} \\
&= u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\Bigg|_{x=2}^{x=3} \\
&= \sec\theta + \frac{1}{2}\ln\left|\frac{\sec\theta-1}{\sec\theta+1}\right|\Bigg|_{x=2}^{x=3} \\
&\quad \textit{(Since } \tan\theta = x,\ \sec\theta = \sqrt{1+x^2}\textit{.)} \\
&= \sqrt{1+x^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|\Bigg|_{x=2}^{x=3} \\
&= \sqrt{10} + \frac{1}{2}\ln\left|\frac{\sqrt{10}-1}{\sqrt{10}+1}\right| - \sqrt{5} - \frac{1}{2}\ln\left|\frac{\sqrt{5}-1}{\sqrt{5}+1}\right| \\
&\approx \boxed{1.07998}
\end{aligned}
$$

- (7.6 #28) Let $Q(t)$ be the number of animals at time t. $Q = 1800$ in 1980, which we'll call $t = 0$; $Q = 2000$ in 1986 ($t = 6$). Population is limited by resources to a max. population of $B = 5000$. Using a logistical model, we have

  $$\frac{dQ}{dt} = kQ(B - Q)$$

  Which leads to the general solution

  $$\begin{aligned} Q(t) &= \frac{B}{1 + Ae^{-Bkt}} \\ &= \frac{5000}{1 + Ae^{-5000kt}} \end{aligned}$$

  where A and k are unknown constants. We'll use the given initial data to determine their values. First will plug in $t = 0$:

  $$\begin{aligned} Q(0) &= \frac{5000}{1 + Ae^{0}} \\ 1800 &= \frac{5000}{1 + A} \\ 1800 + 1800A &= 5000 \\ 1800A &= 3200 \\ A &= \frac{16}{9} \end{aligned}$$

  That gave us the value of A. Now to solve for k we'll plug in the values for $t = 6$:

  $$\begin{aligned} Q(6) &= \frac{5000}{1 + Ae^{-5000(k)(6)}} \\ 2000 &= \frac{5000}{1 + \frac{16}{9}e^{-5000(k)(6)}} \\ 2000 + 2000\left(\frac{16}{9}\right)e^{-30,000k} &= 5000 \\ \frac{2000 \cdot \frac{16}{9}}{3000} &= e^{30,000k} \\ \frac{32}{27} &= e^{30,000k} \\ k &= \frac{\ln\left(\frac{32}{27}\right)}{30,000} \\ &\approx 5.66 \times 10^{-6} \end{aligned}$$

  Now we know the values of the constants. We can determine the value of Q in 2005 ($t = 25$):

  $$\begin{aligned} Q(25) &= \frac{5000}{1 + \left(\frac{16}{9}\right)e^{-5000(5.66\times10^{-6})(25)}} \\ &\approx 2665.42 \end{aligned}$$

  $\boxed{\text{So our logistic model leads to an estimate of about 2665 animals in 2005.}}$

- (7.7 #6)

$$\begin{aligned}\int_1^\infty \frac{dx}{\sqrt{x}} &= \lim_{N\to\infty}\int_1^N \frac{dx}{\sqrt{x}}\\ &= \lim_{N\to\infty} 2\sqrt{x}\Big|_1^N\\ &= \lim_{N\to\infty} 2\sqrt{N}-2\end{aligned}$$

The limit goes to ∞, so the improper integral $\boxed{\text{diverges}}$.

- (7.7 #42)

$$\begin{aligned}\int_0^1 \frac{x\ dx}{1-x^2} &= \lim_{N\to 1^-}\int_0^N \frac{x\ dx}{1-x^2}\\ & \quad \textit{(Let } u=1-x^2,\ du=-2x\ dx\textit{)}\\ &= \lim_{N\to 1^-}\int_{x=0}^{x=N} \frac{-\frac{1}{2}\ du}{u}\\ &= \lim_{N\to 1^-} -\frac{1}{2}\ln|u|\Big|_{x=0}^{x=N}\\ &= \lim_{N\to 1^-} -\frac{1}{2}\ln|1-x^2|\Big|_0^N\\ &= \lim_{N\to 1^-} -\frac{1}{2}\ln|1-N^2|+\frac{1}{2}\ln|1|\\ &= \lim_{N\to 1^-} -\frac{1}{2}\ln|1-N^2|+0\end{aligned}$$

The limit goes to ∞, so the improper integral $\boxed{\text{diverges}}$.

- (7.8 #4)

$$\begin{aligned}\sinh^{-1}(x) &= \ln(x+\sqrt{x^2+1}) \ \textit{for all } x\\ \sinh^{-1}(0) &= \ln(0+\sqrt{1})\\ &= \ln(1)\\ &= \boxed{0}\end{aligned}$$

- (7.8 #14)

$$\begin{aligned}y &= \cosh(1-2x^2)\\ \frac{dy}{dx} &= \sinh(1-2x^2)(-4x)\\ &= \boxed{-4x\sinh(1-2x^2)}\end{aligned}$$

- (7.8 #26)

$$
\begin{aligned}
y &= \sinh^{-1} x - \sqrt{1+x^2} \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1+x^2}} - \frac{1}{2}(1+x^2)^{-1/2}(2x) \\
&= \boxed{\tfrac{1-x}{\sqrt{1+x^2}}}
\end{aligned}
$$

- (7.8 #44)

$$
\begin{aligned}
\int_0^{\ln 2} \sinh(3x)\, dx &= \frac{1}{3}\cosh(3x)\Big|_0^{\ln 2} \\
&= \frac{1}{3}\cosh(3\ln 2) - \frac{1}{3}\cosh(0) \\
&= \frac{1}{3}\left(\frac{e^{3\ln 2} + e^{-3\ln 2}}{2}\right) - \frac{1}{3}\left(\frac{e^0 + e^{-0}}{2}\right) \\
&= \frac{1}{3}\left(\frac{(e^{\ln 2})^{-3} + (e^{\ln 2})^3}{2}\right) - \frac{1}{3}\left(\frac{2}{2}\right) \\
&= \frac{1}{3}\left(\frac{2^3 + 2^{-3}}{2}\right) - \frac{1}{3} \\
&= \frac{1}{3}\left(\frac{2^3 + 2^{-3}}{2}\right) - \frac{1}{3} \\
&= \frac{8}{6} + \frac{1}{48} - \frac{1}{3} \\
&= \frac{49}{48} \\
&\approx \boxed{1.0208}
\end{aligned}
$$