Math 1352-11 — HW02 Solutions

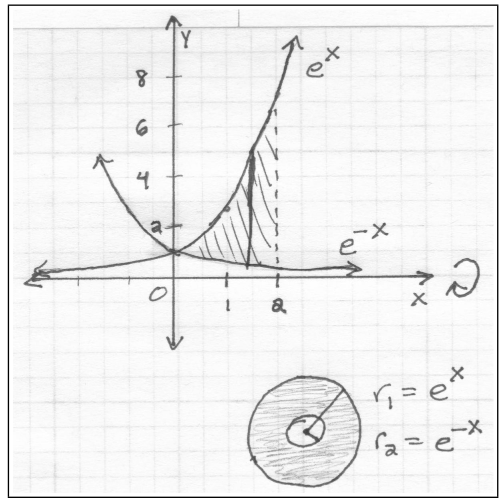
September 11, 2008

Assigned problems: 6.2 – 20, 24, 52; 6.3 – 8, 12, 30, 32, 46

Always read through the solution sets even if your answer was correct.

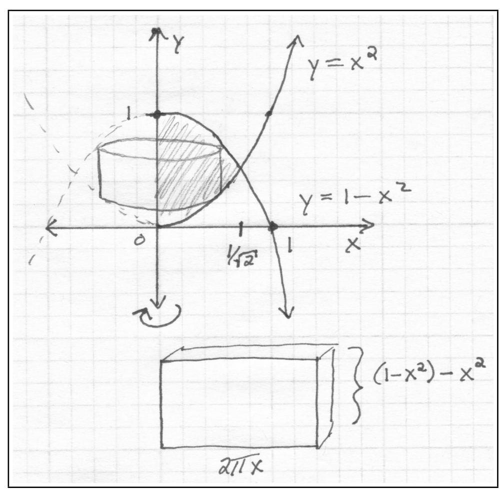
- (6.2 #20) Region bounded by $y = e^x$ and $y = e^{-x}$ on $[0, 2]$ revolved about the x-axis.
  The curves intersect at $x = 0$ and $e^x \geq e^{-x}$ on $[0, 2]$.
  A vertical slice gives a washer with outer radius e^x and inner radius e^{-1}.

$$\begin{aligned} V &= \int_0^2 \pi\left(e^{2x} - e^{-2x}\right)\, dx \\ &= \frac{\pi}{2}\left(e^{2x} - e^{-2x}\right)\Big|_0^2 \\ &= \boxed{\tfrac{\pi}{2}\left(e^4 + e^{-4} - 2\right)} \end{aligned}$$

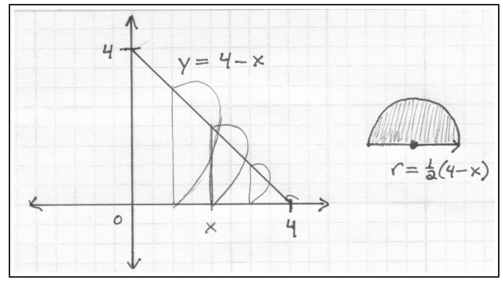


- (6.2 #24) Region bounded by $y = x^2$, $y = 1 - x^2$, and the y-axis for $x \geq 0$ revolved about the y-axis.
  Curves $y = x^2$ and $y = 1 - x^2$ intersect at $x = \pm\frac{1}{\sqrt{2}}$.
  A cylindrical shell at x has circumference $2\pi x$ and height $1 - x^2 - x^2 = 1 - 2x^2$. Total volume is

$$\begin{aligned} V &= \int_0^{\frac{1}{\sqrt{2}}} 2\pi x\left(1 - 2x^2\right)\, dx \\ &= 2\pi \int_0^{\frac{1}{\sqrt{2}}} x - 2x^3\, dx \\ &= \pi\left(x^2 - x^4\right)\Big|_0^{\frac{1}{\sqrt{2}}} \\ &= \boxed{\tfrac{\pi}{4}} \end{aligned}$$



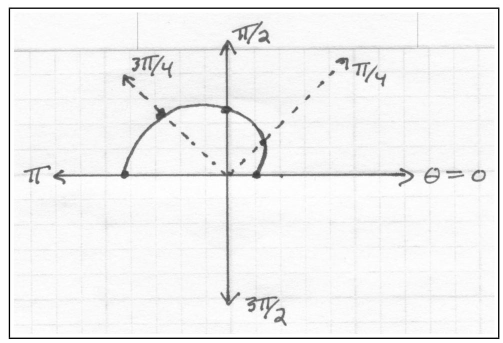
- (6.2 #52) Base is isosceles right triangle with legs of length 4.
  Cross section is a semicircle.
  We'll line up the legs with the x and y axes. The hypotenuse is then given by the line $y = 4 - x$. A semicircle perpendicular to the x-axis has diameter $y = 4 - x$ and radius $r = \frac{1}{2}(4 - x)$. So a semicircle has area $\frac{1}{2}\pi r^2 = \frac{\pi}{8}(4 - x)^2$. The total volume is then



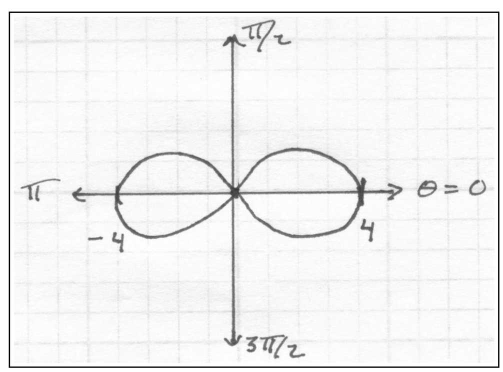
$$
\begin{aligned}
V &= \frac{\pi}{8}\int_0^4 (4-x)^2\,dx \\
&= \frac{\pi}{8}\int_0^4 16 - 8x + x^2\,dx \\
&= \frac{\pi}{8}\left(16x - 4x^2 + \frac{1}{3}x^3\right)\Big|_0^4 \\
&= \boxed{\frac{8\pi}{3}}
\end{aligned}
$$

- (6.3 #8) Graph $r = \theta + 1$.
  We can see that r is growing with θ. Plotting a few points:

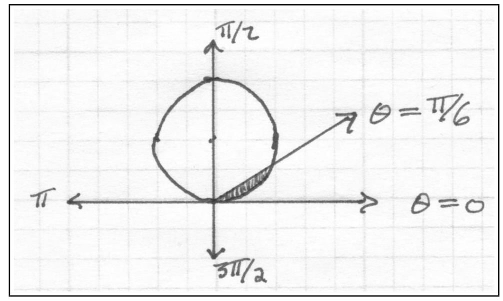
r	θ
0	1
$\pi/4$	$\pi/4 + 1 \approx 1.79$
$\pi/2$	$\pi/2 + 1 \approx 2.57$
$3\pi/4$	$3\pi/4 + 1 \approx 3.35$
π	$\pi + 1 \approx 4.14$



- (6.3 #12) Graph $r^2 = 16\cos 2\theta$
  Comparing to the standard forms, we see that this is a lemniscate in standard form with 2 loops.
  Each loop has length 4.



- (6.3 #30) Area bounded by $f(\theta) = \sin\theta$ on $0 \le \theta \le \pi/6$.
  Area between rays $\theta = 0$ and $\theta = \pi/6$ is

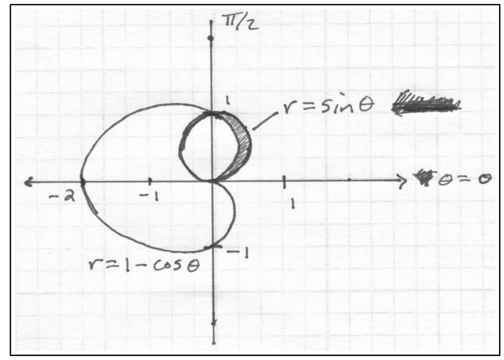


$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\frac{\pi}{6}} r^2 \, d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta \\
&= \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\Bigg|_0^{\frac{\pi}{6}} \\
&= \frac{\pi}{24} - \frac{\sin(\pi/3)}{8} \\
&= \boxed{\tfrac{\pi}{24} - \tfrac{\sqrt{3}}{16}}
\end{aligned}$$

- (6.3 #32) Area bounded by $f(\theta) = \sec\theta$ on $-\frac{\pi}{4} \le \theta \le \pi/4$.

$$\begin{aligned}
A &= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 \, d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2\theta \, d\theta \\
&= \frac{1}{2}\tan\theta\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{1}{2}\left(\tan\left(\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right)\right) \\
&= \frac{1}{2}(1+1) \\
&= \boxed{1}
\end{aligned}$$

- (6.3 #46) Area inside $r = \sin\theta$ and outside $r = 1 - \cos\theta$.
  $r = \sin\theta$ is a rose with one circular petal.
  $r^2 = 1 - \cos\theta$ is a cardioid. The area between them is bounded by the rays $\theta = 0$ and $\theta = \pi/2$.
  The area inside $r = \sin\theta$ on $0 \le \theta \le \pi/2$ is



$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta,$$

and the area inside $r = 1 - \cos\theta$ on $0 \le \theta \le \pi/2$ is

$$A_2 = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos\theta)^2 \, d\theta.$$

So the area between the curves is

$$\begin{aligned}
A_1 - A_2 &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta - (1-\cos\theta)^2 \, d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta - (1 - 2\cos\theta + \cos^2\theta) \, d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin^2\theta - \cos^2\theta) - 1 + 2\cos\theta \, d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} -\cos 2\theta - 1 + 2\cos\theta \, d\theta \\
&= \frac{1}{2}\left(-\frac{1}{2}\sin 2\theta - \theta + 2\sin\theta\right)\Bigg|_0^{\frac{\pi}{2}} \\
&= \boxed{1 - \frac{\pi}{4}}
\end{aligned}$$