Solutions to Final Exam Practice Questions

Math 1351-011

12/4/2007

(1) $\displaystyle\lim_{x\to 1} \frac{x^2-2x+1}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)^2}{(x-2)(x-1)}$

$\displaystyle = \lim_{x\to \phi} \frac{x-1}{x-2} = \frac{0}{-1} = 0$

$\displaystyle\lim_{x\to 2} \frac{\sqrt{x}-2}{x-4} = \frac{\sqrt{2}-2}{2-4} = \frac{\sqrt{2}-2}{-2} = 1-\frac{\sqrt{2}}{2}$

(2) $\displaystyle\lim_{x\to\infty} \frac{\ln(\ln x)}{x}$ ($\frac{\infty}{\infty}$ form; use L'hopital)

$\displaystyle = \lim_{x\to\infty} \frac{\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)}{1}$

$\displaystyle = \lim_{x\to\infty} \frac{1}{x\ln x} = \frac{1}{\infty} = 0$

$\displaystyle\lim_{x\to 0} (\sin x)(\ln x)$ ($0\cdot -\infty$, indeterminate)

$\displaystyle = \lim_{x\to 0} \frac{\ln x}{\csc x}$ (since $\frac{1}{\sin x} = \csc x$)

now $\frac{\infty}{\infty}$ form, so can use L'Hop.

$\displaystyle = \lim_{x\to 0} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0} \frac{-\sin x}{x} \frac{\cos x}{\sin x}$

$\displaystyle = \lim_{x\to 0} \frac{-\cos x}{x} = \frac{-1}{0} = -\infty$

(3) $f(x) = \begin{cases} \dfrac{ax-4}{x-2}, & x \neq 2 \\ b, & x = 2 \end{cases}$

$x=2$ is only suspicious point.
For continuity at $x=2$ we need 3 Things
(1) $f(2)$ defined, (2) $\lim\limits_{x\to 2} f(x)$ exists,

and (3) $\lim\limits_{x\to 2} f(x) = f(2)$.

$f(2) = b$ ✓
So we just need $\lim\limits_{x\to 2} \dfrac{ax-4}{x-2} = b$

Denom. goes to 0 as $x \to 2$, so for limit to exist, we need something in numerator to cancel with $(x-2)$ in denom.
If $a=2$, we have $\dfrac{2x-4}{x-2} = \dfrac{2(x-2)}{x-2}$

$\lim\limits_{x\to 2} \dfrac{2x-4}{x-2} = \lim\limits_{x\to 2} \dfrac{2\cancel{(x-2)}}{\cancel{(x-2)}} = 2.$

Now need this limit to equal $f(2)=b$.
Therefore $b=2$.
$\boxed{a=2,\ b=2}$

(4) $f(x) = x^3 + 2x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 + 2(x+\Delta x) - x^3 - 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + \cancel{2x} + 2\Delta x - \cancel{x^3} - \cancel{2x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + \Delta x^2 + 2$$

$$= 3x^2 + 2$$

(5) $g(x) = \sqrt{x}\cos x + x\cot x$

$$g'(x) = \tfrac{1}{2}x^{-1/2}\cos x + x^{1/2}\cdot(-\sin x) + \cot x - x\csc^2 x$$

$$= \frac{\cos x}{2\sqrt{x}} - \sqrt{x}\sin x + \cot x - x\csc^2 x$$

$$y(x) = \frac{x^2 + \tan x}{3x + 2\tan x}$$

$$y'(x) = \frac{(3x+2\tan x)(2x+\sec^2 x) - (x^2+\tan x)(3+2\sec^2 x)}{(3x+2\tan x)^2}$$

$$= \frac{\begin{pmatrix} 6x^2 + 4x\tan x + 3x\sec^2 x + 2\tan x\sec^2 x \\ - 3x^2 - 3\tan x - 2x^2\sec^2 x - 2\tan x\sec^2 x \end{pmatrix}}{(3x+2\tan x)^2}$$

$$= \frac{3x^2 + (4x-3)\tan x - (2x^2 - 3x)\sec^2 x}{(3x+2\tan x)^2}$$

(5) cont. $h(x) = e^x \tan^{-1} x$

$h'(x) = e^x \tan^{-1} x + e^x \left(\frac{1}{1+x^2}\right)$

(6) $f(x) = \sin(3x^2 + 2x)$ (chain rule)

$f'(x) = \cos(3x^2+2x) \cdot (6x+2)$

ie, let $u = 3x^2+2x$ $f(u) = \sin u$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$$= (\cos u)(6x+2)$$

$$= \cos(3x^2+2x) \cdot (6x+2)$$

$g(t) = e^{-3t^2} \cos 2t + (2t+1)^{10}$

(chain rule, product rule, power rule)

$$g'(t) = (e^{-3t^2})(-2 \sin 2t) + (e^{-3t^2})(-6t)(\cos 2t) + 10(2t+1)^9(2)$$

$$= -2e^{-3t^2} \sin 2t - 6t e^{-3t^2} \cos 2t + 20(2t+1)^9$$

(7) $f(x) = x^2 + 3x + \sin x$, tangent line at $x = 0$

Eqn. of tangent line to graph at $x = x_0$ is given by $y = f'(x_0)(x - x_0) + f(x_0)$

$f(0) = 0$

$f'(x) = 2x + 3 + \cos x$

$f'(0) = 0 + 3 + 1 = 4$

$y = 4(x - 0) + 0$

$\boxed{y = 4x}$

(8) $e^{xy} + y^2 = x^2 + x + 1$

$e^{xy}\left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 2x + 1$

$\left(2y + xe^{xy}\right)\frac{dy}{dx} = 2x + 1 - ye^{xy}$

$$\frac{dy}{dx} = \frac{2x + 1 - ye^{xy}}{2y + xe^{xy}}$$

(9) $x^2+y^2=100$, $\frac{dx}{dt}=10$

what is $\frac{dy}{dt}$ when $y=6$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$
$$2y\frac{dy}{dt}=-2x\frac{dx}{dt}$$
$$\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}$$

when $y=6$

$x^2=100-y^2$

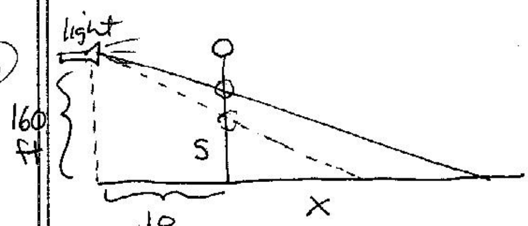
$=100-36=64$

$x=\pm 8$

when $y=6$

$$\frac{dy}{dt}=\frac{-(\pm 8)}{6}\cdot 10=\pm\frac{80}{6}=\pm\frac{40}{3}$$

(10)



x = length of shadow

s = height of ball

$a(t)=-32\ \text{ft}/s^2$

$v(t)=\int -32=-32t+C$ $\quad v(0)=0$ ~~so~~ $C=0$

$v(t)=-32t\ \ (=ds/dt)$

$s(t)=\int v(t)\,dt=-16t^2+C$

initial height $=160$ ft $=s(0)$

$s(0)=0+C=160$, $\quad C=160$

$s(t)=-16t^2+160$

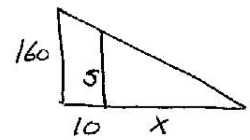
at $t=1$ $\quad v(1)=\left.\frac{ds}{dt}\right|_1=-32\ \text{ft}/s$

$s(1)=-16+160=144$ ft

(10) cont.

By similar triangles

$$\frac{160}{10+x} = \frac{s}{x}$$



$$160x = 10s + xs$$

so at $t=1$, $s(1) = 144$ Therefore

$$160x = 1440 + 144x$$

$$16x = 1440$$

$$x = 90 \text{ ft}$$

$$160x = 10s + xs$$

$$160\frac{dx}{dt} = 10\frac{ds}{dt} + x\frac{ds}{dt} + s\frac{dx}{dt}$$

$$(160 - s)\frac{dx}{dt} = (10 + x)\frac{ds}{dt}$$

$$\frac{dx}{dt} = \frac{(10+x)}{(160-s)}\frac{ds}{dt}$$

at $t=1$, we have $s = 144$, $x = 90$, $\frac{ds}{dt} = -32$

$$= \frac{(10+90)}{(160-144)}(-32)$$

$$= \frac{100}{16}(-32)$$

$$= -200 \text{ ft/s}$$

= velocity of ball's shadow at $t = 1$

(11) $f(x) = x^3 - 3x^2$ on $[-1, 3]$

For abs. extrema on closed bounded interval, need to check critical pts ($f' = 0$ or f' does not exist) and end pts of interval.

$f'(x) = 3x^2 - 6x$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ or 2 = critical pts.

$$\left.\begin{aligned} f(-1) &= -4 \\ f(0) &= 0 \\ f(2) &= -4 \\ f(3) &= 0 \end{aligned}\right\}$$

absolute max at $x = 0$ and $x = 3$

absolute min at $x = -1$ and $x = 2$

(12) $f(x) = \frac{1}{3}x^3 - 9x$

- domain is all real numbers
- $f'(x) = x^2 - 9 = (x+3)(x-3)$

  $f' = 0$ at $x = \pm 3$ (critical pts)

  $f' > 0$ when $|x| > 3$ ie. $x > 3$ or $x < -3$

  (graph increasing)

  $f' < 0$ when $|x| < 3$, ie, $-3 < x < 3$

  (graph decreasing)
- $f''(x) = 2x$

  $f'' = 0$ at $x = 0$ (inflection pt)

  $f'' > 0$ when $x > 0$ (conc. up)

  $f'' < 0$ when $x < 0$ (conc. down)

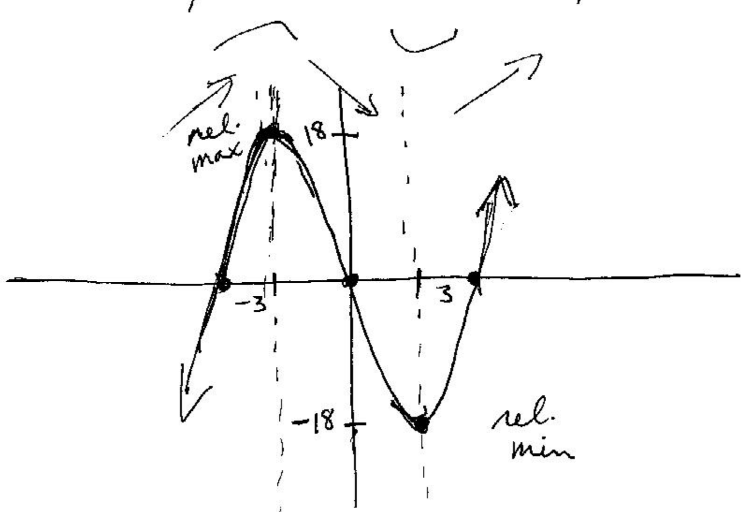
$f(0) = 0 \quad (0,0) \quad$ (y-intercept)

$0 = \frac{1}{3}x^3 - 9x$

$0 = x\left(\frac{1}{3}x^2 - 9\right)$

$x = 0$ or $\pm\sqrt{27}$ (x-intercepts)

at critical pts $f(3) = -18$, $f(-3) = 18$



(13) $f(x) = \dfrac{2x+3}{x-1}$

domain: $(-\infty, 1) \cup (1, \infty)$ $\quad f(0) = \frac{3}{-1} = -3$

range: $(-\infty, \infty)$ $\quad f(x) = 0 \quad x = -3/2$

vert. asymptote at $x = 1$ $\quad$ x-intercept $(-\frac{3}{2}, 0)$

y-intercept $(0, -3)$

$$\lim_{x \to 1^+} \frac{2x+3}{x-1} = \frac{5}{0^+} = \infty$$

$$\lim_{x \to 1^-} \frac{2x+3}{x-1} = \frac{5}{-0} = -\infty$$

horiz. asymptotes:

$$\lim_{x \to \infty} \frac{2x+3}{x-1} = \lim_{x \to \infty} \frac{2}{1} = 2 \quad \text{(L'Hopital)}$$

$$\lim_{x \to -\infty} \frac{2x+3}{x-1} = 2$$

horiz. asy. at $y = 2$

$$f' = \frac{(x-1)(2) - (2x+3)(1)}{(x-1)^2} = \frac{-5}{(x-1)^2}$$

$f' \neq 0$ (no rel. max or min)

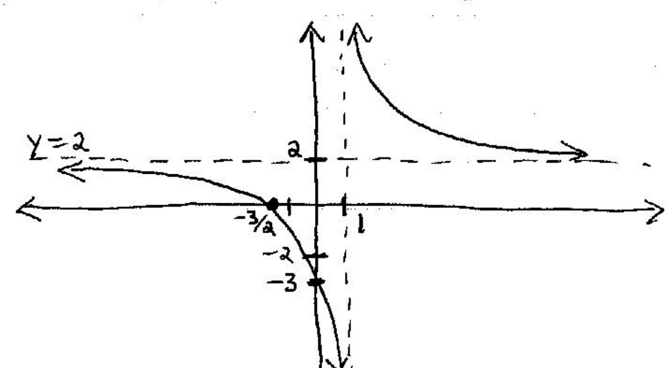
$f' < 0$ for all x (graph always decreasing)

$f'' = 10(x-1)^{-3} = 10/(x-1)^3$

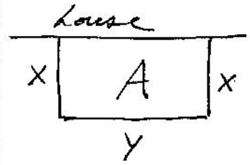
$f'' \neq 0$ (no inflection pts)

$f'' > 0$ when $x - 1 > 0$, $x > 1$ (conc. up)

$f'' < 0$ when $x - 1 < 0$, $x < 1$ (conc. down)



(14)



$$2x + y = 100$$
$$y = 100 - 2x$$

$$A = xy$$
$$= x(100 - 2x)$$
$$= 100x - 2x^2$$
$$A' = 100 - 4x$$
$A' = 0$ when $100 = 4x$
$$x = 25$$
$A'' = -4 < 0$ (conc. down, so $x = 25$ is a relative max.)

$$y = 100 - 2 \cdot 25 = 50$$

max area is $25 \cdot 50 = 1250 \text{ ft}^2$

(15) $\int (x^2+1)^2\,dx = \int (x^4 + 2x^2 + 1)\,dx$
$$= \frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C$$

$$\int 3\cos x + \frac{2}{\sqrt{1-x^2}}\,dx = 3\int \cos x\,dx + 2\int \frac{dx}{\sqrt{1-x^2}}$$
$$= 3\sin x + 2\sin^{-1} x + C$$

16 $f(x) = x^3 + 3x \qquad \int_0^1 (x^3 + 3x)\,dx$

Choose regular intervals and right end pts:

$\Delta x = \frac{b-a}{n} = \frac{1}{n}$ for $n = 4$ $\quad \Delta x = \frac{1}{4}$

$$f(k\Delta x) = f(k/4) = \frac{k^3}{4^3} + \frac{3k}{4}$$

$$S_4 = \sum_{k=1}^{4} \left(\frac{k^3}{4^3} + \frac{3k}{4}\right)\left(\frac{1}{4}\right)$$

$$= \sum_{k=1}^{4} \frac{k^3}{4^4} + \frac{3k}{16} \qquad 4^4 = 16 \cdot 16 = 256$$

$$= \left(\frac{1}{256} + \frac{3}{16}\right) + \left(\frac{8}{256} + \frac{6}{16}\right) + \left(\frac{27}{256} + \frac{9}{16}\right) + \left(\frac{64}{256} + \frac{12}{16}\right)$$

$$= \frac{100}{256} + \frac{30}{16} = \frac{100}{256} + \frac{480}{256} = \frac{580}{256}$$

$$= 145/64 \approx \int_0^1 (x^3 + 3x)\,dx$$

17 $\int_0^1 (2x+1)^4\,dx \qquad$ let $u = 2x + 1$

$du = 2\,dx \quad \left(dx = \frac{1}{2}du\right)$

$u(0) = 1, \quad u(1) = 3$

$$= \int_1^3 \frac{1}{2} u^4\,du$$

$$= \frac{1}{2} \cdot \frac{1}{5} u^5 \Big|_1^3 = \frac{1}{10} u^5 \Big|_1^3$$

$$= \frac{1}{10}(3)^5 - \frac{1}{10}(1)^5$$

$$= \frac{243}{10} - \frac{1}{10} = \frac{242}{10} = 121/5$$

(17) cont. $\int \sqrt{3x+2}\, dx$

let $u = 3x+2$
$du = 3\, dx$
$(dx = \frac{1}{3}\, du)$

$= \int \frac{1}{3} u^{1/2}\, du$

$= \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C$

$= \frac{2}{9} u^{3/2} + C$

$= \frac{2}{9} (3x+2)^{3/2} + C$

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

let $u = x^{1/2}$
$du = \frac{1}{2} x^{-1/2}\, dx$
ie., $\frac{1}{\sqrt{x}}\, dx = 2\, du$

$= \int 2 e^{u}\, du$

$= 2e^{u} + C$

$= 2e^{\sqrt{x}} + C$

$\int \frac{\ln x}{x}\, dx$

let $u = \ln x$
$du = \frac{1}{x}\, dx$

$= \int u\, du$

$= \frac{1}{2} u^{2} + C$

$= \frac{1}{2} (\ln x)^{2} + C$

(18) $y = x^2 + x + 1$ on $[-2, 2]$

$y \geq 0$ on $[-2, 2]$
Therefore area

$$A = \int_{-2}^{2} (x^2 + x + 1)\, dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x \Big|_{-2}^{2}$$

$$= \left(\frac{1}{3}\cdot 8 + \frac{1}{2}\cdot 4 + 2\right) - \left(\frac{1}{3}\cdot(-8) + \frac{1}{2}\cdot 4 - 2\right)$$

$$= \frac{16}{3} + 4 = \frac{16}{3} + \frac{12}{3}$$

$$= \boxed{28/3}$$

$y' = 2x + 1$
$= 0$ at $x = -1/2$
Can check abs. max, min:
$y(-1/2) = 3/4$
$y(-2) = 3$
$y(2) = 7$
So $y(-1/2) = 3/4$ is the abs. min on $[-2, 2]$.
Therefore $y \geq 3/4$ on $[-2, 2]$