Announcements

- Homework 12 due Friday, 11/30
- Bonus homework due Wednesday, 12/5
- Homework 13 due Wednesday, 12/5
- Final exam: Monday, 12/10, 10:00 to 12:00, BA 057.
  - (Optional) review session Friday, 12/6, at 1:00 in CH 107
  - I'll have practice problems as usual. You can also buy old finals (and solutions) from the math club. See fliers in math building.

Riemann Sums

Suppose a bounded function f is given along with a closed interval $[a, b]$ on which f is defined.

1. Partition the interval $[a, b]$ into n subintervals by choosing points $[x_0, x_1, \ldots, x_n]$ arranged so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

   Call this partition P.
   For $k = 1, 2, 3, \ldots, n$, the k^{th} subinterval width is $\Delta x_k = x_k - x_{k-1}$. The largest of these widths is called the **norm** of the partition P and is denoted $\|P\|$.

$$\|P\| = \max_{k=1,2,\ldots,n} \{\Delta x_k\}$$

2. Choose a number x_k^* arbitrarily from each subinterval $[x_{k-1}, x_k]$. This number is called the k^{th} **subinterval representative** of the partition P.

3. Form the sum

$$\begin{aligned} R_n &= f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n \\ &= \sum_{k+1}^{n} f(x_k^*)\Delta x_k \end{aligned}$$

   This is the **Riemann sum** associated with f, the given partition P and the chosen subinterval representatives $x_1^*, x_2^*, \ldots, x_n^*$.

Note that the Riemann sum does not require that the function be nonnegative.

Definite Integral

If f is defined on the closed interval $[a, b]$, we say that f is **integrable on** $[a, b]$ if

$$I = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists. (x_k^* = subinterval representatives.)

This limit is called the **definite integral** of f from a to b. The definite integral is denoted by

$$I = \int_a^b f(x)\, dx$$

Definite integral at a point:

$$\int_a^a f(x)\,dx = 0$$

Interchanging the limits of a definite integral:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Theorem: If f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$.

Area as an integral: Suppose f is continuous and $f(x) \geq 0$ on the closed interval $[a, b]$. Then the area under the curve $y = f(x)$ on $[a, b]$ is given by the definite integral of f on $[a, b]$. I.e.,

$$\text{AREA} = \int_a^b f(x)\,dx$$

If g is cont. and $g \leq 0$ on the closed interval, then $-g(x) \geq 0$ and

$$\text{AREA} = -\int_a^b g(x)\,dx$$

If f is cont. and sometimes positive and sometimes negative, then

$$\int_a^b f(x)\,dx = A_1 - A_2$$

- A_1 is the sum of all areas of regions above the x-axis and below the graph of f (i.e., where $f(x) \geq 0$) and
- A_2 is the sum of all areas of the regions below the x-axis and above the graph of f (i.e., where $f(x) \leq 0$).

Properties of the definite integral

- **Linearity:** If f and g are integrable on $[a, b]$, then so is $rf + sg$, for constants r and s.

$$\int_a^b [rf(x) + sg(x)]\,dx = r\int_a^b f(x)\,dx + s\int_a^b g(x)\,dx$$

- **Dominance:** If f and g are integrable on $[a, b]$ and $f(x) \le g(x)$ throughout the interval, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

- **Subdivision:** For any number c such that $a < c < b$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Distance as an integral

The **total distance traveled** by an object with continuous velocity $v(t)$ along a straight line from time $t = a$ to $t = b$ is

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} |v(a + (k-1)\Delta t)|\Delta t = \int_a^b |v(t)|\, dt$$